6th International Symposium on Generalized Convexity/Monotonicity

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ABSTRACTS OF TALKS

In all abstracts the name of the speaker appears first.

INVITED TALKS

Set-Valued Optimization

Johanes JAHN

Nowadays set-valued optimization means set-valued analysis and its application to optimization, and it is an extension of continuous optimization to the set-valued case.

First we discuss set-valued optimization problems and present solution concepts. In the second part we investigate contingent epiderivatives of set-valued maps. Properties of this differentiability notion, a possible generalization and simple optimality conditions are presented.

The third part is devoted to subgradients. Two possible generalizations of the well-known notion in convex analysis are considered: subgradients and weak subgradients. Properties and simple optimality conditions are given for these two concepts.

The Lagrange multiplier rule is the topic of the fourth part. For a constrained optimization problem with set-valued maps a necessary optimality condition is shown and a suitable regularity assumption is investigated. Generalized quasiconvex set-valued maps are introduced and a sufficient optimality condition is presented.

Although set-valued optimization is a rapidly growing field of research there are still open questions being discussed at the end of this talk.

Minimization of the Sum of Several Linear Fractional Functions

Hiroshi KONNO

This paper is concerned with global minimization algorithms for solving rank p linear fractional programming problems, namely the minimization of the sum of $p(> 1)$ linear fractional functions over a polytope. Algorithms to be discussed are: a parametric simplex algorithm for rank two problems; a convergent approximate algorithm for rank three problems; a generalized convex multiplicative programming algorithm and a piecewise convex underestimation/ branch and bound algorithm. We will show that we are able to obtain a globally optimal solution for up to rank eight problems in a practical amount of time.

Global continuous approaches to discrete optimization problems

Panos M. PARDALOS

Discrete (or combinatorial) optimization problems, that is, problems with a discrete feasible domain and/or a discrete domain objective function, model a large spectrum of applications in computer science, operations research and engineering.

Solution methods for discrete optimization problems can be classified into combinatorial and continuous approaches. A typical combinatorial approach generates a sequence of states, which represent a partial solution, drawn from a discrete finite set. Continuous approaches for solving discrete optimization problems are based on different equivalent characterizations in a continuous space. These characterizations include equivalent continuous formulations, or continuous relaxations (including semidefinite programming), that is, embeddings of the discrete domain in a larger continuous space.

There are many ways to formulate discrete problems as equivalent continuous problems or to embed the discrete feasible domain in a larger continuous space (relaxation). The surprising variety of continuous approaches reveal interesting theoretical properties which can be explored to develop new algorithms for computing (sub)optimal solutions to discrete optimization problems.

We are going to discuss continuous approaches to several discrete problems, including the maximum clique problem, graph coloring, the satisfiability problem, and many minimax problems (e.g. sphere packing).

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Discrete Higher Order Convex Functions and Their Applications

András PRÉKOPA

The purpose of the paper is to survey recent results obtained in connection with higher order convex functions and discrete moment problems and present a number of applications of the relevant theories.

Given a discrete function $f(z)$, $z \in Z = \{z_0, \dots, z_n\}$, the first order divided differences are defined as $\frac{f(z_i)-f(z_{i-1})}{z_i-z_{i-1}}$ and the higher order divided differences are defined iteratively. The function f is said to be convex of order m, if its all $(m+1)$st order devided differences are positive. Given a subset $\{z_i, i \in I\}$ of Z with $|I| = m+1$, the corresponding $(m+1)$st order divided difference is the fraction of two determinants, where in the denominator we have the Vandermonde determinant $\left| (z_i^\alpha)_{i\in I}^{\alpha=0,\dots,m} \right|$ and if we replace its last row by $(f_i, i \in I)$, we obtain the determinant in the numerator.

Discrete higher order convex functions play important role in discrete power moment problems

$$\min(\max) \sum_{i=0}^{n} f(z_i)p_i$$

subject to

$$\sum_{i=0}^{n} z_i^k p_i = \mu_i, \quad i = 0, \dots, n,$$
$$p_i \geq 0, \quad i = 0, \dots, n,$$

where the unknowns are the probabilities $p_0, \dots, p_n$ while known are: the support Z and the first m moments $\mu_1, \dots, \mu_m$ of the probability distribution. If we specialize Z as the set $\{0, \dots, n\}$, replace z_i^k by $\binom{i}{k}$ and μ_i by S_i, where S_i is the ith binomial moment of the random variable, then the resulting LP is called the binomial moment problem. Both the disrcete power and binomial moment problems have been thoroughly studied in the past few years. The structures of the dual feasible bases are fully described and elegant dual type algorithms are presented for the solution of the problems.

The discrete moment problems are generalized in two different directions: (a) the powers/binomial coefficients are replaced by elements of a Chebyshev system (b) the problems are formulated for the multivariate case. Both (a) and (b) involve interesting generalized convex functions. In addition to the theoretical and algorithmic developments we present applications to bounding probablities, expectations, communication and transportation system reliability, valuation of options, among others.

CONTRIBUTED TALKS

Support curves ensuring the trigonometrical convexity

Mohamed S. ALI

Trigonometrically convex functions have interesting applications in the theory of entire functions and in the theory of cavitational diagrams for hydroprofiles (see.[1],[2]).

The aim of this work is to obtain a geometric characteristic for trigonometrically convex functions, namely, an analogue of the known theorem for ordinary convex functions, which states that:

A function $f : (a, b) \to \mathbf{R}$ is convex if and only if there is at least one line of support for f at each $x_0 \in (a, b)$.

Definition. A function $T(\alpha) = a \cos \alpha + b \sin \alpha$ is called a support for π-periodic function f, which is defined on the real line $\mathbf{R}$ at point α_0, if

$T(\alpha_0) = f(\alpha_0)$, and $T(\alpha) \leq f(\alpha)$ for all α.

For trigonometrically convex functions, Bonsall [3] proved the existence of a support function. In the present paper we show the Sufficiency condition: if there exist a support function for a π-periodic function f at each point $\alpha_0 \in \mathbf{R}$, then f is trigonometrically convex.

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Normal operator in quasiconvex analysis: monotonicity, regularity and integration properties

Didier AUSSEL and Aris DANIILIDIS

We define the "normal operator" of a function f as a point-to-set map which associates to any point x of the domain of the function the normal cone at x to the sublevel set $S_{f(x)}$. The concept of normal cone considered here is an abstract one thus covering a large number of situations.

The normal operator appears to be an efficient tool in quasiconvex analysis since it allows to characterize different classes of quasiconvexity under weak regularity assumptions.

Regularity and "integration" properties of the normal operator are considered. In particular we characterize classes of functions having the same normal operator.

On Mixed Lagrangian function and Saddle point Optimality Criteria in Mathematical Programming

C. R. BECTOR and Suresh CHANDRA

A MIXED Lagrange function is used to study saddle point optimality criteria for a class of nonlinear programming problem under generalized convexity assumptions. This study is further extended to certain class of fractional and generalized fractional programming problems as well.

Generalized Rho-Invexity and Duality for Nonsmooth Multiobjective Continuous Programming Problems

Davinder BHATIA

The consepts of nonsmooth Rho-invexity and generalized nonsmooth Rho-invexity for the continuous functions are introduced. These concepts are then utilized to establish duality results for the multiobjective continuous programming problems involving nondifferentiable Lipschitz functions.

Saddlepoint Optimality Criteria in Vector Optimization

Giancarlo BIGI

In this paper, Lagrangian functions for a rather general vector optimization problem are discussed. Mainly, two different approaches can be considered: the first one is based on vector Lagrangians and concepts of saddlepoint for vector valued functions, the other one consists in considering a scalar Lagrangian also for vector optimization problems and relies on the ordinary concept of saddlepoint. In particular, we study the relationships between the saddlepoints of these types of Lagrangians and we propose a unified approach. In this unified framework, we present saddlepoint optimality criteria, which generalize the known ones.

First and Second Order Characterizations of a Class of Pseudoconcave Vector Valued Functions

Riccardo CAMBINI and Laura MARTEIN

In these very last years, several classes of vector valued pseudoconcave functions have been introduced and studied with the aim of extending to the vector case some properties of scalar pseudoconcavity [1-11].

In this paper we deeply analyze a class of vector valued pseudoconcave functions, among the whole proposed ones, which extends to the vector case both pseudoconcavity and strictly pseudoconcavity as well as their optimality properties, such as the global optimality of local optima, of critical points and of points verifying Kuhn-Tucker conditions.

This class of functions, introduced in [4,6], comes out to be particularly relevant since it is possible for it to determine both first and second order characterizations.

The proposed class of vector valued pseudoconcave function thus offers a complete extension to the vector case of the well known scalar pseudoconcavity, giving the chance to work in multiobjective optimization with all the properties of the scalar case.

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On the supremum of some classes of Nonlinear Fractional Problems

Laura CAROSI , Alberto CAMBINI and Laura MARTEIN

The optimization problems with unbounded feasible regions have been handled in many papers of the recent literature. Theoretical and algorithmic aspects has been studied in order to find both constructive new methods and conditions under which the maximum value exists. See for example the survey given by Auslander [1] or the wide literature of fractional programming problem dealing with polyhedral feasible region (Schaible [2]).

Even though we can find many results for optimality conditions, there is almost nothing about the supremum of a function over an unbounded feasible region. On the other hand, when a function does not attain maximum value it is important to know if its supremum is finite or not. In this paper, for the remarkable role played both in optimization and in economic theory, we turn our attention to a nonlinear fractional problem where the objective function is a ratio between a quadratic and a linear function and the feasible region is any closed and unbounded set. Since conditions which ensure a finite supremum are not easy to reach for the general case, we limit ourselves to consider the case where the quadratic function is convex or concave, or it is the product between two affine functions. For these classes of problems, necessary and/or sufficient conditions for a finite supremum are established by means of the recession cone of the feasible set and suitable directions associated with such a cone. Our results cover the linear and the linear fractional case which can be seen as a particular case of the previous ones.

References

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Existence Results For Equilibrium Problems and Applications

Ouyal CHADLI and H. RIAHI

In this paper, we are interested to equilibrium problems in an other point of view, more precisely we show an existence result for equilibrium problems in a noncoercive framework without using Ky Fan Theorem. We give some applications to scalar and vector optimization, to existence of Pareto Optima and to variational-hemivariational inequalities.

The Steiner Ratio of L_p^3-spaces

Dietmar CIESLIK and Jens ALBRECHT

We consider Steiner's Problem in $\mathcal{L}_p^d$, which is a d-dimensional space equiped with p-norm. Steiner's Problem is the "Problem of shortest connectivity", that means, given a finite set N of points in the plane, search for a network interconnecting these points with minimal length. This shortest network must be a tree and is called a Steiner Minimal Tree (SMT). It may contain vertices different from the points which are to be connected. Such points are called Steiner points.
If we do not allow Steiner points, that means, we only connect certain pairs of the given points, we get a tree which is called a Minimum Spanning Tree (MST) for N.
Steiner's Problem is very hard as well in combinatorial as in computational sense, but on the other hand, the determination of an MST is simple. Consequently, we are interested in the greatest lower bound for the ratio between the lengths of these both trees:

$$m(d,p) := \inf \left\{ \frac{L(\text{SMT for } N)}{L(\text{MST for } N)} : N \subseteq \mathcal{L}_p^d \text{ is a finite set} \right\},$$

which is called the Steiner ratio (of $\mathcal{L}_p^d$).
We present the knowledge of $m(d,p)$ for several quantities of d and p. Moreover, we look for estimates for $m(3,p)$, and we will determine general upper bounds for the Steiner ratio of $\mathcal{L}_p^3$, depending on the value of p.

Global Invexity

Bruce D. CRAVEN

Global invexity is characterized by a condition which is independent of the scale function describing the invexity. Consequently, weak duality holds for the Wolfe, or Mond-Weir, dual problem when a sufficient invex condition is replaced by a suitable inequality condition. This holds exactly when the Wolfe dual is equivalent to the Lagrangian dual. Results are given for differentiable, and for locally Lipschitz, functions.

Recovering an utility function from a differentiable demand function

Jean-Pierre CROUZEIX and Tamás RAPCSAK

Assume that the behaviour of a consumer can be described through an utility function u, that is to say, the consumer determines his choice by maximizing u on the commodity set G subject to a budget constraint. Assume that G is the nonnegative orthant, $\hat{p} \geq 0$ is the vector of prices and $w > 0$ is the income of the consumer, the problem is:

$$\text{maximize } u(x) \text{ subject to } x \geq 0,\ \ \hat{p}^t x \leq w.$$

Set $p = \frac{\hat{p}}{w}$ then the consumer's problem becomes

$$v(p) = \max[\, u(x) \ :\ x \geq 0,\ p^t x \leq 1].$$

We denote by $X(p)$ the set of optimal solutions of this problem. The map X is called the demand correspondence, the function v is called the indirect utility function associated to u. Under some reasonable conditions, u can be recovered from v via the minimization problem:

$$u(x) = \min[\, v(p) \ :\ p \geq 0,\ p^t x \leq 1].$$

In fact, the concept of utility is rather theorical since the observations of the behaviour of a consumer give his choices (the sets $X(p)$, i.e. the demand correspondence) when faced with a normalized vector of prices but not a representation in terms of an utility function. Furthermore, if an utility exists, it is not uniquely defined since, given an utility function u and k an increasing and continuous function of one real variable, the function $\hat{u}(x) = k(u(x))$ describes the behaviour of the consumer as well. The real problem is to construct, when possible, an utility function from the knowledge of the demand correspondence.

Since the early begining of the theory of consumer, this question has been the object of a special attention (the pionneers are Samuelson and Houthakker). It is known in economics as the problem of *revealed preferences*. For a mathematician, it is analogous to the problem of constructing a convex function f from a monotone map F in such a way that the subgradient of f coincides with F. However, the problem here is quite more complicated because in the convex problem the function f, if it exists, is uniquely determined up to an additive constant while in our case the utility function u will be determined up to a scalarisation function k.

This paper is concerned with the case where the demand correspondance X is single-valued, i.e. $X(p) = \{x(p)\}$. We assume also that x is negative and continuously differentiable on the positive orthant. We shall show how to construct a differentiable indirect utility function v corresponding to x. The study is based on a concept of cyclically pseudomonotonicity which is closely related to the axioms of revealed preferences introduced by Samuelson and Houthakker. The case where X is multivalued is the object of a current joint work with A. Eberhard.

On the question of integration of multivalued operators

Aris DANIILIDIS, Mohamed BACHIR and Jean-Paul PENOT

Rockafellar's integration technique for cyclically monotone operators is extended into a more general setting: The Fenchel-Moreau subdifferential is replaced by the lower subdifferential (introduced by Plastria) and a more general class of integrable multivalued operators arises.

Fixed Point Theorems, Coincidence Theorems, and Variational Inequalities

Behzad DJAFARI ROUHANI

We give a simple proof to an extension of a fixed point theorem of E.Tarafdar for multivalued mappings and show its equivalence to a KKM type result.A noncompact coincidence point theorem is also established. Applications of these results to establish the existence of solutions to variational inequalities in not necessarily reflexive Banach spaces are also considered.

Some remarks on one function and two functions minimax theorems

András DOMOKOS

We will analyse various types of generalized concave-convexities which are used in minimax theorems. Some of them constitute enough strong assumptions for one function, but not for two functions minimax theorems for example downward-upward defined in the paper of S. Simons. We will study the difference between the assumptions of these minimax theorems.

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Subjets of DC Functions and DC Optimality

Andrew EBERHARD

Formulae for the approximate, Michel-Penot and Clarke subdifferential of the difference of two functions have been derived by numerous authors. These usually involve special kinds of set-differences and in some cases only provide outer approximations for these subdifferentials. One of the motivations for interest in such formulae is in the potential application they have to optimality conditions for DC programming (difference of convex functions). Clearly first-order conditions will only be neccessary and one will require second-order information in order to provide sufficient optimality conditions. A formula for a second-order subdifferential of a difference of functions would allow unconstrained optimality criteria to be applied to the DC programming model and leave open the potential of obtaining sufficient optimality conditions. Progress in this direction will be discussed. Generalized convexity arguments are used to establish such a formula for the Subjet of a difference of functions.

Decomposition of a polynomial function as a difference of convex functions. Application to a model of hydro-electric generation

Albert FERRER BIOSCA

When we use specific techniques of global optimization for solving a nonconvex optimization problem, it often happens that the functions may be written as a difference of convex functions. This representation is said to be a d.c. decomposition and the functions are said to be d.c.functions. The class of d.c.functions displays a remarkable stability with respect to operations frequently encountered in optimization. The main purpose when using these properties is how to construct a d.c. decomposition of a function which is known to be a d.c.function but not given in d.c. form. In the present report we show a method to obtain a d.c. decomposition for polynomial functions. We also use this method with the polynomial functions of a model of hydro-electric generation.

Simplified Optimality Conditions for Minimizing a Difference of Vector-Valued Functions

Fabian FLORES-BAZÁN and Werner OETTLI

Given an arbitrary non-empty set X, we consider the minimization problem

$$\text{(P)} \qquad \min\{g(x) - h(x) | x \in X\},$$

where $g, h : X \to Z \cup \{+\infty\}$ are proper and Z is a topological vector space endowed with an ordering given by a convex cone $P \subseteq Z$ such that $\operatorname{int} P \neq \emptyset$ and $P \cap (-P) = \{0\}$. Thus, the ordering introduced in Z is reflexive, transitive, and antisymmetric, i.e., $(z \geq 0$ and $z \leq 0) \implies z = 0$. We define furthermore $z_1 > z_2$ (equivalently $z_2 < z_1$) iff $z_1 - z_2 \in \operatorname{int} P$. It is easily seen that $z_1 \geq 0$ and $z_2 > 0$ implies $z_1 + z_2 > 0$.

The crucial order theoretic assumption we have to make is that $(Z, \geq)$ is *order-complete*. This means that every nonempty subset of Z, which has an upper bound, also has a supremum.

Finally we adjoin to Z an artificial element, $+\infty$ say, such that $+\infty \geq z$ and $+\infty + z = z + (+\infty) = +\infty$ for all $z \in Z \cup \{+\infty\}$. Every nonempty subset of Z, bounded or not, has a supremum in $Z \cup \{+\infty\}$.

For an arbitrary function $f : X \to Z \cup \{+\infty\}$,

$$\operatorname{dom} f := \{x \in X \mid f(x) \in Z\},$$

and f is called proper iff $\operatorname{dom} f \neq \emptyset$.

In addition, we fix Φ, a nonempty family of functions defined on X with values in Z. For $\overline{x} \in \operatorname{dom} f$ and $\varepsilon \in Z$ with $\varepsilon \geq 0$ we define

$$\partial_\varepsilon f(\overline{x}) := \{\varphi \in \Phi \mid f(x) - f(\overline{x}) \geq \varphi(x) - \varphi(\overline{x}) - \varepsilon \ \forall\, x \in X\}.$$

It is the ε-subdifferential of f at $\overline{x}$ with regard to the family Φ. Clearly $\partial_{\varepsilon_1} f(\overline{x}) \subseteq \partial_{\varepsilon_2} f(\overline{x})$ if $\varepsilon_1 \leq \varepsilon_2$.

In dealing with extended real-valued functions, minorants are often more convenient than subgradients. Accordingly, for $\overline{x} \in X$ and $\alpha \in Z$ with $\alpha \leq f(\overline{x})$ we define

$$\mu_\alpha f(\overline{x}) := \{\varphi \in \Phi \mid f(x) \geq \alpha + \varphi(x) - \varphi(\overline{x}) \ \forall\, x \in X\},$$

the α-minorants of f at $\overline{x}$ with regard to Φ. Clearly, if $f(\overline{x}) \in Z$, then $\mu_\alpha f(\overline{x}) = \partial_\varepsilon f(\overline{x})$ whenever $\alpha + \varepsilon = f(\overline{x})$.

We say that $x^0 \in \operatorname{dom} g \cap \operatorname{dom} h$ is a solution of (P) iff

$$g(x) - g(x^0) \geq h(x) - h(x^0) \ \forall\, x \in X.$$

In connection with problem (P) we shall have occasion to use the following assumptions about the function h:

(H1) for all $\overline{x} \in X$, if $\alpha \in Z$ and $\alpha < h(\overline{x})$, then $\mu_\alpha h(\overline{x}) \neq \emptyset$;

(H2) for all $\overline{x} \in X$, if $\alpha \in Z$ and $\alpha \leq h(\overline{x})$, then $\mu_\alpha h(\overline{x}) \neq \emptyset$.

Clearly, (H2) implies (H1). If $\overline{x} \in \operatorname{dom} h$, then (H1) implies that $\partial_\varepsilon h(\overline{x}) \neq \emptyset \ \forall \varepsilon > 0$, and (H2) implies that $\partial_0 h(\overline{x}) \neq \emptyset$. Moreover, if $h : X \to Z$ and $\partial_0 h(x) \neq \emptyset \ \forall x \in X$ then h satisfies (H2). We propose to call h "almost Φ-supported", iff (H1) holds, and "Φ-supported", iff (H2) holds. We will quote various alternative situations for the space X, Z, and Φ, besides h, where assumption (H1) or (H2) is satisfied. In addition, we will establish simplified optimality conditions regarding assumption (H2) along with duality results. Finally several sufficient conditions ensuring the validity of our simplified optimality condition are also discussed.

Subdifferentials of Nonsmooth Functions and Second Order Optimality Conditions

Ivan GINCHEV

Arbitrary nonsmooth function $f : \mathbf{E} \to \overline{\mathbb{R}}$ is considered, here $\mathbf{E}$ is a Banach space and $\overline{\mathbb{R}}$ is the set of the reals extended with the infinite elements $\pm\infty$. The approximation of f near a point $x_0 \in \mathbf{E}$ with quadratic functionals is studied and described with the defined in the paper lower directional derivatives. In terms of these derivatives second order optimality conditions are derived and a characterization of the isolated minimizers of order 2 is found. Differentials, subgradients and subdifferentials of order $n = 0, 1, 2$, consistent with the directional derivatives are defined. The lower second order Taylor function $T_-^{(2)} f(x_0, \cdot)$ is introduced and similarly to the smooth case it is discussed in which sense $T_-^{(2)} f(x_0, v)$ approximates near x_0 the function f with accuracy $o(\|v\|^2)$, the respective property in the paper is referred as $E_-^{(2)}$ property. Representation of $T_-^{(2)} f(x_0, \cdot)$

with subdifferentials is discussed, and the existence of such representation is explained in terms of 2-convexity. Finally the second order optimality conditions are reformulated in terms of the introduced subdifferentials of order $n = 0, 1, 2$.

Proper Efficiency and Generalized Convexity in Nonsmooth Vector Optimization Problems

Giorgio GIORGI and Angelo GUERRAGGIO

In vector optimization problems several notions of proper efficiency have been proposed, in order to rule out some situations (tolerated by the definition of efficiency) scarcely meaningful. The relationships between the above different definitions have been studied; in this paper we again show these relationships, under the assumption that the involved functions are not differentiable but only Lipshitzian.

The properties of Clarke's generalized derivative and subdifferential are here used, in order to get a picture similar to the one obtained for the differentiable case. Then, by means of suitable generalized convexity assuptions, we obtain some equivalences of the different notions of proper efficiency considered, with respect to the notion of efficiency. So the results of the paper may be viewed as a generalization of the situation established for the smooth case.

Duality for Fractional MinMax Problems Involving Arcwise Connected and Generalized Arcwise Connected Functions

Pankaj GUPTA and Davinder BHATIA

In this paper, we introduce two duals for the fractional minmax programming problem:

$$minimize \underset{y \in Y}{maximize} \frac{f(x,y)}{h(x,y)}$$

$$\text{subject to } g(x) \leq 0,$$

and establish duality results under arcwise connectedness and generalized arcwise connectedness assumptions on the functions involved.

Generalized convexity for unbounded sets: the enlarged affine space

Guillermo HANSEN

As is well known, a closed convex set in $\mathbb{R}_n$ is unbounded if and only if it contains a half-line and every parallel from any of its points. This fact suggest the idea, already pointed out by R.T. Rockafellar in his classical book on Convex Analysis, and recently developed to some extent by Rockafellar and Wets in their book Variational Analysis, of considering the directions defined by such half-lines as a kind of "improper" or "infinity" points, and the half-lines themselves as segments with a proper and an improper endpoint. In the "enlarged" affine space so defined convexity means, as usual, that the segment joining any two points of a set is contained in the set. A theory of convex sets in the enlarged affine space, parallel to the classical one, is developed here, including "enlarged" versions of separation theorems, combinatorial theorems, support theorems and Minkowski's theorem on extreme points.

Solving Fuzzy Inequalities with Concave Membership Functions

Cheng-Feng HU and S.-C. FANG

Solving systems of fuzzy inequalities could lead to the solutions of fuzzy mathematical programs. It is shown that a system of fuzzy inequalities with concave membership functions can be converted to a regular convex programming problem. A "method of centres" with "entropic regularization" techniques is proposed for solving such a problem. Some computational results are included.

A Note on Minty Variational Inequalities and Generalized Monotonicity

Reinhard JOHN

Some notions of generalized monotonicity for multi-valued mappings are characterized in terms of properties of the associated Minty variational inequalities. In particular, it is shown that the Minty variational inequality problem derived from a map F defined on a convex domain is solvable on any nonempty, compact, and convex subdomain if and only if F is properly quasimonotone.

On dual representation of generalized convex functions

Gábor KASSAY

In this paper we discuss several classes of generalized convex functions introduced in the literature and their relationship.

For a convex cone $K \subset \mathbf{R}^{\mathbf{m}}$, we investigate four vector valued function classes: *K-convex*, *K-convexlike*, *K-subconvexlike* and *closely K-convexlike* functions. Using the dual representation for closed convex cones (the bipolar theorem), and the dual representation of the relative interior of a convex cone, we give a complete characterization for K-subconvexlike and closely K-convexlike functions in terms of real valued functions.

Finally we discuss Lagrangean duals and their relation with the above class of generalized convex vector valued functions.

Optimality Conditions in Nonsmooth Multiobjective Programming

Do Sang KIM

In this talk, we give a counterexample of some results in Majumdar [JOTA 92(1997), 419-427]. We consider the following nonsmooth multiobjective programming problem involving locally Lipschitz functions:

$$\begin{array}{lll} \text{(MP)} & \text{Minimize} & f(x) = (f_1(x), \cdots, f_m(x)) \\ & \text{subject to} & g(x) = (g_1(x), \cdots, g_p(x)) \leqq 0, \\ & & h(x) = (h_1(x), \cdots, h_q(x)) = 0, \\ & & x \in X(\subset R^n),\ X \text{ is open.} \end{array}$$

We obtain the generalized Karush-Kuhn-Tucker conditions for (MP). We prove optimality theorems for (weak) Pareto-optimal solutions of (MP) under generalized convexity assumptions.

Sharply pseudoconvex functions

Iosif KOLUMBAN

Let X be a real Banach space and D be a convex subset of X. A function $f : D \to \mathbf{R}$ is called *sharply convex*, if there exists some $c > 0$ such that for each $x, y \in D$ and $0 \le t \le 1$ one has

$$\langle x - y, \nabla f(y)\rangle \ge 0 \implies f(x) \ge f(x + t(y - x)) + ct(1 - t)\|x - y\|^2.$$

It is obvious that each sharply pseudoconvex function is also pseudoconvex. Each strongly convex function is also sharply pseudoconvex, but not vise-versa.

Proposition. Let D be an open convex subset of the real Banach space X, and let $f : D \to \mathbf{R}$ be a given differentiable function. Then f is sharply pseudoconvex if and only if the gradient map ∇f is strongly pseudomonotone.

Example. The function $f : [0, 1] \to \mathbf{R}$ given by $f(x) = \ln(1 + x)$ is sharply quasiconvex.

Let W be a topological space, let X be a reflexive Banach space, Y a normed space and let C and D be nonempty closed convex subsets of X and Y, respectively. Denote by $(X, Y)^*$ the set of all linear, continuous mappings defined on X which take values in Y. Consider the function $f : W \times X \to \mathbf{R}$ and the mappings $L : W \to (X, Y)^*$ and $a : W \to Y$. For each $w \in W$, take the parametric optimization problem

$(\mathrm{P})_{\mathrm{w}}$ $$\min\{f(w, x) \mid x \in C,\ a(w) + L(w)(x) \in D\}.$$

Denote by $K(w)$ the feasible set of problem $(\mathrm{P})_{\mathrm{w}}$, i.e. the set

$$\{x \in C \mid a(w) + L(w)(x) \in D\}.$$

In the following we suppose that this feasible set is nonempty for each $w \in W$.

Let $x_0 \in K(w_0)$ be the unique solution of the initial problem $(\mathrm{P})_{\mathrm{w_0}}$. Then this (initial) problem is said to be *stable under perturbations* if there exist a neighbourhood W_0 of w_0 and a mapping $x : W_0 \to X$ continuous at w_0 such that $x(w_0) = x_0$ and $x(w)$ is a solution of problem $(\mathrm{P})_{\mathrm{w}}$ for each $w \in W_0$.

For a certain $w \in W$, the feasible set $K(w)$ is called *regular*, if

$$0 \in \mathrm{int}\{a(w) + L(w)(x) - y \mid x \in C,\ y \in D\}.$$

Suppose the functions $f(w, .) : X \to \mathbf{R}$ are differentiable for each $w \in W$ and denote by ∇f the gradient map of f with respect to its second variable. Then the functions $f(w, .) : X \to \mathbf{R}$ are said to be *uniformly sharp pseudoconvex* on the subset $V \subset W$, if there exists a constant $c > 0$ such that for all $w \in V$ and $x, y \in X$, $0 \le t \le 1$ one has

$$\langle x - y, \nabla f(w, y)\rangle \ge 0 \quad \implies \quad f(w, x) \ge f(w, x + t(y - x)) + ct(1 - t)\|x - y\|^2.$$

Now we can state the stability result for problem $(\mathrm{P})_{\mathrm{w_0}}$.

Theorem. Suppose that $K(w_0)$ is regular and

(i') x_0 is the unique solution of $(\mathrm{P})_{\mathrm{w_0}}$;

(ii') The mapping $(w, x) \mapsto \nabla f(w, x)$ is continuous at (w_0, x_0);

(iii') There exists a neighbourhood V of w_0 such that the functions $f(w, .)$ are uniformly sharp pseudoconvex on V, and for each $w \in V$ the map $\nabla f(w, .)$ is continuous from the line segments of X to the weak topology of X^*.

Then problem $(\mathrm{P})_{\mathrm{w_0}}$ is stable under perturbations.

Convex-, D-convex and quasiconvex Farkas Theorems

Sándor KOMLÓSI

Gyula (Julius) Farkas investigating the stable equilibrium of a mechanical system proved a dual characterization of the Fourier Principle based on a result on linear inequality systems [1, 2]. This result has been quoted since the publishing of [3] the seminal paper of Kuhn and Tucker on Nonlinear Programming as Farkas Lemma. András Prékopa pointed to the fact in his papers [4, 5] devoted to the early history of the Optimization Theory that Farkas' merit was not only to have provided a rigorous proof of a result on linear inequality system, but the invention of the Karush-Kuhn-Tucker necessary optimality condition itself (in case of the regularity had been taken granted).

The aim of the present talk is to give a summary on nonlinear extensions of Farkas' result including convex, D-convex (difference of convex) and quasiconvex generalizations of the Farkas Lemma.

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On Vector Equilibrium and Vector Variational Inequality Problems

Igor V. KONNOV

The scalar equilibrium problem has numerous applications in Mathematical Physics, Economics and Game Theory and includes optimization and variational inequality problems. The vector equilibrium problem (in short, VEP) is a generalization of the scalar one and it is also extensively investigated. In this work we consider the relationships between VEP and other general vector problems. First we investigate several ways of transforming VEP into a vector variational inequality problem under the K-space setting and obtain the relationships between monotonicity properties of the corresponding functions. Next, we present a new gap function for VEP which allows one to reduce VEP into a vector optimization problem with single-valued function. We also consider applications of these results to vector saddle point and vector inverse optimization problems.

A Remark on Implicit Vector Variational Inequalities

Sangho KUM

In a previous work the authors proved some existence results of solutions of implicit vector variational inequalities (IVVI) for compact valued multifunctions under generalized weak pseudomonotonicity assumptions and Hausdorff topological vector space setting. In this talk, following the approach of Konnov and Yao, we investigate the existence of solutions of IVVI for noncompact valued multifunctions in the same situation. Furthermore, we provide an existence theorem of IVVI for multifunctions with convex values and open lower sections without the generalized pseudomonotonicity.

Vector Invex n-set Functions and MinMax Programming

Promila KUMAR and **Davinder BHATIA**

In this paper, we introduce vector invexity and generalized vector invexity for n-set functions and utilize these definitions to establish sufficient optimality and duality results for a class of minmax programming problems involving n-set functions. Applications of these results to fractional programming problem are also presented.

Some properties of a.e. generalized convex functions

Davide LA TORRE and **Matteo ROCCA**

The notion of almost convex functions was introduced by M.Kuczma in 1970 (see [8]) for real mid-convex functions to study some classes of functional equations. In the work Kuczma studied some properties of mid-convex functions and gave an affirmative answer to the following question: is an almost mid-convex function almost everywhere (a.e.) equal to a mid-convex function? In the following years (1986) Parnami and Vasudeva introduced a notion of almost convex function and δ-almost convex function for real function of real variable. In the paper Parnami and Vasudeva proved that "near" to an almost δ-convex function there is a convex function and the difference between these functions is less or equal to 3δ for almost everywhere $x \in (a, b)$. As corollary of this result they proved that an almost convex function is a.e. equal to a convex function. These studies arose as generalization, for convex functions, of the corresponding problem for additive functions (i.e. for real functions of real variable which satisfy the Cauchy's functional equation: $f(x+y) = f(x) + f(y)$) and almost additive functions. In the talk we will give characterizations of a.e. generalized convex functions ($f : \mathbf{R^n} \to \mathbf{R}$) by using the Sobolev spaces and weak derivatives and some applications of these functions in optimization problems. At the end we will show an application of a.e. generalized convex functions in stochastic optimization problems.

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Fractional Complex Programming Problem Involving Generalized Convex Analytic Functions

Hang-Chin LAI

We consider a complex nonlinear fractional programming:

$$(\mathrm{P}) \qquad \min \ \frac{Re\ [f(z,\bar{z}) + (z^H Az)^{1/2}]}{Re\ [g(z,\bar{z}) - (z^H Bz)^{1/2}]} \equiv \frac{Re\ F(z,\bar{z})}{Re\ G(z,\bar{z})}$$
$$\text{s. t.} \quad z \in C^n \ \text{ and } \ h(z,\bar{z}) \in S$$

where $f,\ g : C^n \times C^n \to C$, and $h : C^n \times C^n \to C^m$ are analytic functions, $A,\ B \in C^{n\times n}$ are positive semidefinite Hermitian matrices, and S is a clsoed convex cone in C^m. We establish the optimality conditions for problem (P) involving generalized $(\mathcal{L},\rho,\theta)$-convexity for functions F, G and h. Employing the optimality conditions, we construct a dual model to problem (P) and establish the weak/strong duality theorems.

Generalized stochastic convexity and stochastic orderings of mixtures

Claude LEFEVRE, M. DENUIT and S. UTEV

In this work, a new concept called generalized stochastic convexity is introduced as an extension of the classical notion of stochastic convexity. It relies on the well-known concept of generalized convex functions, and corresponds to a stochastic convexity with respect to some Tchebycheff system of functions. A special case discussed in detail is the notion of stochastic s-convexity ($s \in I\!N$), which is obtained when this system is the family of power functions $\{x^0, x^1, \ldots, x^{s-1}\}$. The analysis is made, firstly for totally positive families of distributions, and then for families that do not enjoy that property.

Further, integral stochastic orderings, said of Tchebycheff-type, are introduced that are induced by cones of generalized convex functions. For s-convex functions, they reduce to the s-convex stochastic orderings studied recently. These orderings are then used for comparing mixtures and compound sums, with some illustrations in epidemic theory and actuarial sciences.

Existence of Equilibria for Multivalued Mappings and Its Applications to Vectorial Equilibria

Lai-Jiu LIN and Zenn-Tsuen YU

Let X be a convex subset in a real topological vector space E, $\varphi : X \times X \to R$ a given function with $\varphi(x, x) = 0$ for all $x \in X$. By an equilibrium problem (shortly EP), Blum and Oettli understood the problem of finding

$$x \in X \text{ such that } \varphi(\overline{x}, y) \leq 0 \text{ for all } y \in X.$$

This problem contains the optimization problem, problem of the Nash type equilibria, complementarity problems, fixed point problems, variational inequality problems and many others as special case.

Moreover, some variations or generalization of this problem can be possible. Recently, Oettli considered the multivalued vectorial equilibrium problem. He replaced the range R by a real topological vector space Z with an ordering cone P (meaning $P \neq Z$ is a closed convex cone with nonempty interior), and he considered a multimap $\varphi : X \times X \multimap Z$. Then the inequalities $\varphi(x, y) \geq 0$ may be generalized in several possible ways, for instance as $\varphi(x, y) \subset P$, $\varphi(x, y) \cap P \neq \emptyset$, $\varphi(x, y) \cap (-intP) = \emptyset$, $\varphi(x, y) \not\subseteq -intP$, and similarly for $\varphi(x, y) \leq 0$. He give a more general approach. He considered the problem of finding

$$\overline{x} \in X \text{ such that } \varphi(\overline{x}, y) \subset C(\overline{x}) \text{ for all } x \in X, \qquad (*)$$

where $\varphi : X \times X \multimap Z$ and $C : X \multimap Z$.

This problem contains the multivalued variational inequality problem (Lin et al.) of finding

$$\overline{x} \in X \text{ such that } \langle \Phi(\overline{x}), y - \overline{x} \rangle \not\subset C(\overline{x}) \text{ for all } y \in X,$$

where $\Phi : X \multimap L(E, Z)$, and $L(E, Z)$ is the linear space of linear operators from E to Z and $C : X \multimap Z$ is a multimap.

In this paper, we continue to study the existence theorems of (*). We apply a new fixed point theorem of Lin and Yu and use various g-monotonic conditions and some coercivity condition to establish the existence theorems of (*). We generalize some results of Oettli and Schläger. Oettli and Schläger use a KKM theorem due to Fan to establish the existence of (*), but our approach is different from Oettli and Schläger. As a simple consequences of our results, we give a unified approach to vector equilibrium of single-valued functions. Hence our approach to vectorial equilibrium of single-valued functions is quite different from Oettli.

Duality in d.c. programming: the case of several d.c. constraints

Juan Enrique MARTÍNEZ-LEGAZ and Michel VOLLE

The aim of this paper is to provide a duality theory associated with the problem of minimizing the difference of two convex functions (that is a d.c. function) under finitely many d.c. constraints. The possible reduction of such problem to the case of only one d.c. constraint has been discussed by Lemaire and Volle in [1] (see also [2] for a suitable generalization). We will take here another way. Let us consider the general d.c. programming problem

$$(\mathcal{P}) \quad \text{minimize } g(x) - h(x) \quad \text{subject to} \quad g_i(x) - h_i(x) \leq 0 \;,\; i = 1, \cdots, m,$$

where g, h, g_i, h_i are extended real-valued convex functions on a real Hausdorff locally convex topological vector space X. We adopt the conventions

$$(+\infty) - (+\infty) = (-\infty) - (-\infty) = (+\infty) + (-\infty) = (-\infty) + (+\infty) = +\infty$$

and the related calculus rules ([3]).

Assuming that the functions h_i are subdifferentiable on the feasible set of $(\mathcal{P})$ (what does not require in fact the convexity of the $h_i's$ on the whole space), we introduce a dual problem which is entirely expressed in terms of the Legendre-Fenchel conjugates of the data functions. We first establish a weak duality relation without the convexity of $g, g_1, \cdots, g_m$. In the case when these functions are convex and additional qualification conditions are satisfied, we obtain a zero duality gap. Neither these qualification conditions nor the subdifferentiability of the $h_i's$ are necessary if one considers the problem

$$(Q) \quad \text{minimize } g(x) - h(x) \quad \text{subject to} \quad g_i(x) - h_i(x) < 0 \ , \ i = 1, \cdots, m.$$

When only one constraint occurs in (Q) we recapture a result by Lemaire and Volle ([1], Theorem 3.1).Several particular cases of problem $(\mathcal{P})$ are also examined.

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Vector Optimization Under Parameter Dependence

Piera MAZZOLENI

When considering economic inequalities, such as poverty and income inequalities, and the whole family of financial dominance properties we can give an analytical representation for order relations. Monotonocity and concavity properties have been stated. But what is even more interesting, the order relations have led to classes of vector optimization problems, according to the preference set linked with the economic underlying properties.

In correspondence vector optimization problems, characterizing families of efficient solutions, reveal themselves of practical usefulness also for other classes of optimization problems.

This approach of representing the order relation analytically is typical of fuzzy relations where the numbers themselves are represented by functions depending on parameters. This is an immediate way to generate parametric families of vector optimization problems, which can be applied both to fields which are natural for operations research such as graph problems, but also to finance where vagueness characterizes for instance the definition of interest rates, loan demand and deposit supply.

A unified approach is proposed in this paper based on the image set, to state simultaneously monotonicity and concavity properties, not only of the optimal value function, but especially of the optimal solution.

Lagrangian Duality for Multiobjective B-vex Programming with n-set Functions

Aparna MEHRA and Davinder BHATIA

By introducing a modified Lagrangian function, we establish saddle point duality for a class of multiobjective programming problems involving differentiable n-set b-vex functions. Further, using the concept of proper efficiency, Lagrangian type duality theory has been developed for such programming problems.

A Probabilistic Approach to Higher Order Convexity

Sandra MENDONÇA and Dinis D. PESTANA

Let M_n^* denote the class of bounded functions $f \in C^n$ defined on $[0, +\infty]$ such that for $x > 0$, its n-th order derivative $f^{(n)}(x)$ is monotone; we shall call M_n^* the class of monotone functions of order n in $[0, +\infty]$. If for all $a > 0$, $f_a(x) = f(a + x) \in M_n^*$, we say that $f \in M_n$ the class of monotone functions of order n in $(0, +\infty]$.

Let D_{n+1} denote the set of distribution functions $F(x)$ such that i) $f(x) = F'(x)$ exists (except possibly at 0); ii) both $f(x)$, for $x > 0$, and $f(-x)$, for $x < 0$ are monotone functions of order n in $(0, +\infty]$. The probability density functions of distributions with positive support in D_{n+1} provide a natural base for the pointed convex cone of monotone functions of order n in $(0, +\infty]$. By identifying the extremal rays of this well-capped cone we obtain a Choquet-type integral representation for higher order monotone functions. The fact that completely monotone functions are Laplace transforms of non-decreasing functions, and the integral characterization of Stieltjes transforms, are simple corollaries.

Inversion theorems are provided, and an alternative characterization of monotone functions of order n is used, together with fractional integrals, to extend the concept of higher order monotonicity to fractional orders.

It is then possible to define extended unimodality, and to present an unified approach to beta-transformed random variables (and to explore it in connection with order statistics), Pólya and generalized Pólya characteristic functions.

Using the concept of higher order monotonicity, we characterize classes of weak limit laws of maxima of suitably normed independent (but not necessarily identically distributed) random variables. We discuss the relevance of distribution functions of the form $F(x) = e^{-K(x)}$ with $K(x)$ convex (Mejzler's M class) or $K(x)$ completely monotone (the M_∞ class) in modelling extremal events.

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Stochastic Orders generated by generalized convex functions

Alfred MÜLLER

Stochastic orders are an important tool for the comparison of different stochastic models. They appear in a variety of applications including statistics, operations research, economics, epidemiology, ... Many of them are defined as follows:

We say that for two random variables (or random vectors) $\mathbf{X}$ and $\mathbf{Y}$ the ordering $\mathbf{X} \leq_{\mathfrak{F}} \mathbf{Y}$ holds, if $Ef(\mathbf{X}) \leq Ef(\mathbf{Y})$ for all functions f in some class of functions $\mathfrak{F}$. There are several applications where the functions of the class $\mathfrak{F}$ exhibit some properties of generalized convexity, like directional convexity, quasi-convexity, higher-order convexity etc. In our talk we show some situations where such orderings arise, and we show how properties of the orderings are connected with properties of the classes of functions. We present some new results as well as some open problems. (Some parts of the talk are based on joint work with Marco Scarsini.)

Geometric Separation Theorems for Convex Sets

Zsolt PÁLES

One of the most basic forms of the separation theorems for convex sets is due to M. H. Stone. This result states that *if A and B are two disjoint convex sets of a linear space X, then there exist complementary convex sets C, D (that is $C = X \setminus D$) such that $A \subset C$, $B \subset D$.* It is well known that then most of the important separation or sandwich theorems can be derived from this principle.

It seems to be interesting to consider the following related problems:

(P1) Is the statement still true if convexity is replaced by midpoint convexity? The answer to this question is no in general; the characterization of the situation when separability is possible yields interesting consequences for midpoint quasiconvex and quasiconcave functions.

(P2) If A and B are invariant by a family of affine transformations, can one find C and D such that they also share the same invariance property? The answer to this question is yes, if we assume that the given transformations commute with each other. As application, we can obtain sandwich and separation theorems where the separating functionals admit further properties.

(P3) If K is a convex cone and $A + K$ and $B - K$ are disjoint, can one find C, D such that they also have this property? The answer to this question is yes. Applications can be found to find sandwich theorems for K-convex and K-concave functions.

Convexity and generalized convexity methods for the study of Hamilton-Jacobi equations

Jean-Paul PENOT and Michel VOLLE

We present a survey of recent results about the explicit solution of the first-order Hamilton-Jacobi equation. We use formulas of the Hopf and Lax-Oleinik types, but in the quasiconvex case the usual Fenchel conjugacy is replaced by quasiconvex conjugacies known from some years or the usual inf-convolution is replaced by a sublevel convolution. Inasmuch we use weak generalized convexity and continuity assumptions, some of our results are new; in particular, we do not assume that the data are finite-valued, so that equations derived from attainability or obstacle problems can be considered.

Convexity and Decomposability in Multivalued Analysis

Adrian PETRUŞEL and Ghiocel MOŢ

Consider $(T, \mathcal{A}, \mu)$ a complete σ-dinite and nonatomic measure space. If E is a Banach space, let $L^1(T, E)$ be the Banach space of all measurable functions $u : T \to E$ that are Bochner μ-integrable.

We call a set $K \subset L^1(T, E)$ decomposable, iff for all $u, v \in K$ and each $A \in \mathcal{A}$, we have

$$u\chi_A + v\chi_{T\setminus A} \in K,$$

where χ_A stands for the characteristic function of the set A. This notion is, somehow, similar to convexity, but there exist also major differences.

However in several cases the decomposability condition is a good substitute for convexity (see [1], [2], [3] etc.).

The purpose of this paper is to report several theorems and to present other results in the field of multivalued analysis, related to this subject: convexity replaced by decomposability. Some open questions are also pointed out.

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Scalar Characterizations of (Γ, Ω)-quasiconvex Functions

Nicolae POPOVICI

Various generalizations of the classical notion of quasiconvexity of real-valued functions have been given for vector-valued functions, their importance in nonconvex vector optimization being nowadays recognized. Among them, the cone-quasiconvex functions in the sense of Dinh The Luc are of special interest because they possess the characteristic property to have convex level sets. In the well-known Dinh The Luc's monograph on Vector Optimization, it was shown that if E_2 is a real topological vector space, partially ordered by a convex cone C having a nonempty interior, then a function $f : X \to E_2$ is C-quasiconvex on a nonempty convex set $X \subset E_1$ if and only if for any point $a \in E_2$, the composite function $h_{e,a} \circ f : X \to R$ is quasiconvex in the usual sense on X, where $h_{e,a} : E_2 \to R$ denotes the smallest strictly monotonic function defined for an arbitrary fixed point $e \in \operatorname{int} C$.

The main purpose of this paper is to show that even if the algebraic and topological structures of the domain and the codomain are replaced with ordinal structures endowed with an abstract convexity given by a multifunction Γ and a binary relation Ω, some similar characterizations in terms of scalar quasiconvex functions can be given for the general class of so called (Γ, Ω)-quasiconvex functions, which was previously introduced by us in order to describe in a unifying way those functions which possess the characteristic property to have convex level sets. However, in order to characterize the (Γ, Ω)-quasiconvexity in terms of scalar quasiconvex functions, we cannot utilize the smallest strictly monotonic functions in this general setting. For this aim, we introduce a special class of real-valued functions, namely the λ-characteristic functions associated to the binary relation Ω, which can be construct by a simply algebraic approach.

Vector Stochastic Optimization Problems

Giovanna REDAELLI

Many economic and financial problems can be formulated through a decision model with multiple and conflicting objectives; furthermore, the uncertainty of the economic and financial environment induce stochasticity in the definition of the objective and constraint functions. When concerned with the definition of portfolio management problems, usually what really matters is to limit losses and, given the randomness of the problem, it is usually asked to minimize the probability that losses exceed a given value. In such a context it is often useful to describe the problem at hand in terms of a stochastic programming model, which is an optimization model where the functions involved are random variables defined on a given probability space. The objective functions and the constraints may be defined in many different ways according to the problem at hand. In particular, we consider problems of stochastic programming with a convex feasible set and assume as (vector or scalar) objective function what it is known as a loss function, i.e.: given a probability space (Ω, Σ, P), a vector $\phi = [\phi_1 \phi_2 ... \phi_n] \in R^n$ $(n \geq 1)$ and a vector valued function $F = [F_1 F_2 ... F_n]$ $(F_i : R^m \times \Omega \to R$, $i = 1, 2, ..., n)$ we define

$$P_\phi(u) = \begin{matrix} P(\omega \in \Omega : F_1(u,\omega) \leq \phi_1) \\ P(\omega \in \Omega : F_2(u,\omega) \leq \phi_2) \\ : \\ : \\ P(\omega \in \Omega : F_n(u,\omega) \leq \phi_n) \end{matrix} \qquad (1)$$

in the vector case and

$$P_\phi(u) = P(\omega \in \Omega : F_i(u,\omega) \leq \phi_i, i = 1, 2 \ldots n) \qquad (2)$$

in the scalar case, to be the probability that losses don't exceed a prespecified level ϕ. We consider the following optimization problem:

$$\max_{u \in U} P_{\phi}(u) \tag{3}$$

with $P_{\phi}(u)$ defined as in (1) or (2) and study the relationships between the two formulations of the objective function in terms of the optimal solutions and optimal values. To this end, we use well known results from the vector optimization and scalarization methods in order to relate the solutions of problem (3) with objective function (1) to those of the same problem but with objective function (2).

$C^{k,1}$ functions and divided differences

Matteo ROCCA and **Davide LA TORRE**

In this paper we give some necessary and sufficient conditions for a real function of real variables to be of class $C^{k,1}$, that is k times differentiable with locally Lipschitz k-th derivative.

The class of $C^{k,1}$ function has been studied since the work of Hiriart-Urruty, Strodiot and Hien Nguyen [7] who introduced the concept of generalized hessian matrix for $C^{1,1}$ functions proving also second order optimality conditions for nonlinear constrained problems. Later, Luc [10], considering the class of $C^{k,1}$ functions, extended Taylor's formula, proved higher order optimality conditions when derivatives of order greater than k do not exist and provided characterizations of generalized convex functions. The necessary and sufficient conditions mentioned above involve the (local) boundedness of the (k+1)-th divided difference or of the (k+1)-th lower and upper Riemann derivatives. From these results we deduce a Taylor's formula for $C^{k,1}$ functions.
A well-known class of functions with bounded (k+1)-th divided difference is that of (k+1) convex functions. Our results can be also wiewed as generalizations of differentiability properties of (k+1)-convex functions.

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Duality for a non-differentiable programming problem with generalized invexity

Norma G. RUEDA and Shashi K. MISHRA

A general dual to a non-differentiable programming problem is considered and duality results are obtained under weak invexity assumptions. A second- order dual is also presented and duality theorems proved under second-order generalized invexity assumptions.

Positive Sub-Definite Matrices, Generalized Monotonicity and Linear Complementarity Problems

Siegfried SCHAIBLE, Jean-Pierre Crouzeix, Abdelhak Hassouni and L.Lahlou

Positive sub-definite matrices were introduced by Martos to characterize generalized convex quadratic functions. This concept is extended to nonsymmetric matrices. It leads to a study of pseudomonotone matrices and to new characterizations of generalized monotone affine maps. In addition some properties of linear complementarity problems involving such maps are derived.

Duality for Equilibrium Problems under Generalized Monotonicity

Siegfried SCHAIBLE and Igor KONNOV

Duality is studied for an abstract equilibrium problem which includes optimization problems and variational inequality problems, among other classical problems. Following different schemes, various duals are proposed and primal-dual relationships are established under certain generalized convexity and generalized monotonicity assumptions. In a primal-dual setting, existence results for a solution are derived for different generalized monotone equilibrium problems within each scheme.

On suprema of abstract convex and quasi-convex hulls

Ivan SINGER

We show that the (pointwise) supremum of two arbitrary hull operators on a complete lattice is again a hull operator, which can be expressed in a natural way in terms of the two initial hull operators. Our results yield, in particular, very simple proofs of some results of Attéia and ElQortobi on projective bipolars of sets and functions. Also, we give a new way of recuperating abstract convex hulls of sets from abstract quasi-convex hulls of their indicator functions.

Hyper-Sensitivity Analysis via Composite Concave Programming

Moshe SNIEDOVICH

In contrast to conventional sensitivity analysis where a problem is subjected to changes in numeric parameters, hyper-sensitivity analysis deals with structural changes in the objective function. In this presentation we shall examine a number of approaches capable of facilitating such an analysis with special focus on composite concave programming. The basic ideas are illustrated in the framework of composite linear and quadratic programming problems.

Optimality conditions for multiobjective programming problems with generalized d-type-I and related n-set functions

Ion M. STANCU-MINASIAN and Vasile PREDA

We establish some optimality conditions under generalized convexity assumptions for a multiobjective programming problem involving generalized d-type-I and related n-set functions.

Convexity beyond vector spaces, alternative theorems and minimax equality

Anton STEFANESCU

The concept of convexlike (concavelike) functions was introduced by Ky Fan (1953), who has proved the first minimax theorem without linear structure of the underlying spaces. Further extensions or generalizations of this concept have been used later in optimization and decision theory, but the most significant applications are in the framework of the game theory, where the strategy spaces are not endowed with natural algebraic structures.

In the present paper one introduces new convexity and connectedness conditions and establishes the relationships with other known convexlike type properties. The main results concern the minimax equality in a topological framework. They generalize classical minimax theorems of Fan and König, and are independent of most similar results known in the literature. The proofs make use of some special alternative theorems which also hold in a pure topological framework, without any vector space structure.

New invexity-type conditions in constrained optimization

Anton STEFANESCU, Giuseppe CARISTI and Massimiliano FERRARA

As it is known, invexity has been introduced in optimization theory as a substitute of convexity. Since any differentiable scalar function is invex if and only if every stationary point is a global minimizer, the concept of invexity may appear as a general condition. However, in constrained optimization one requires that the objective and the constraint functions are invex with respect to the same vector function. As it was shown first by D.H.Martin (1985), in this context, the invexity is an unnecessarily strong condition for Kuhn-Tucker sufficiency. Several authors considered possible relaxations of the invexity in order to obtain necessary and sufficient conditions for optimality criteria or for weak duality.

In the present paper we define weaker invexity-type properties and examine the relationships between the new concepts and other similar conditions. One obtains on this way necessary and sufficient conditions for Kuhn-Tucker sufficiency. Moreover one proves that the same conditions are sufficient for Wolfe duality.

Numerical Methods for Solving Some Special Max-Min Programming Problems Involving Generalized Convex Functions

Stefan TIGAN, I.M. STANCU-MINASIAN and Iliana TIGAN

The general max-min problem under consideration is one of the form:
MP. Find

$$V = \max_{x \in X} \min_{y \in T(x)} G(x, y)$$

where $X \subset \mathbf{R}^n$, and $Y \subset \mathbf{R}^m$ are two given sets, T is a given point to set function from X to Y and $G : X \times Y \to \mathbf{R}$ is a real-valued function.

This problem has a important application in modeling various conflict situations. Max-min problem MP can be reforulated as a min-max problem if $G(x, y)$ is replaced by $-G(x, y)$.

Min-max and max-min fractional problems arise in the formulation of continuous rational approximation problems with respect to the Chebyshev norm, in continuous rational games, in multi-objective programming, in engineering design and in some parameter estimation problems as well as in some portfolio selection problems. One can add to these the application of max-min problems to some minimum-risk problems.

In this paper we deal with three classes of max-min programming problems. Firstly we refer at some max-min bicriterion fractional programming problems with separate polyhedral constraints. We show that these problems can be transformed into bicriterion fractional maximization problems. The second class involves some polynomial max-min pseudo-monotonic problems with separate polyhedral constraints. The third class comprises some quasi-monotonic max-min problems with joint linear constraints. For each of these classes of max-min problems we propose some specific numerical methods to find an optimal solution.

Minimal and Reduced Pairs of Convex Bounded Sets

Ryszard URBAŃSKI

Let $X = (X, \tau)$ be a topological vector space over the field $\mathbb{R}$. Let $\mathcal{B}(X)$ be a family of all nonempty closed bounded convex subsets of X. For any $A, B \subset X$ the Minkowski sum is defined by $A + B = \{a + b \mid \ a \in A \ \text{ and } \ b \in B\}$. Since $A + B$ is not generaly closed we define $A \overset{*}{+} B = \overline{A + B}$

for $A, B \in \mathcal{B}(X)$. It was showed that for $A, B, C \in \mathcal{B}(X)$ the inclusion $A \overset{*}{+} B \subset B \overset{*}{+} C$ implies $A \subset C$. From this it follows that $\mathcal{B}(X)$ together with "$\overset{*}{+}$" is a semigroup satisfying the law of cancellation, i.e. $A \overset{*}{+} B = B \overset{*}{+} C$ implies $A = C$.

A relation "$\sim$" on $\mathcal{B}^2(X)$ is given by $(A, B) \sim (C, D)$ iff $A \overset{*}{+} D = B \overset{*}{+} C$ and a partial ordering by the relation: $(A, B) \leq (C, D)$ iff $A \subset C$ and $B \subset D$.

The relation "$\sim$" is an equivalence relation in $\mathcal{B}^2(X)$ and "$\leq$" is an ordering in the equivalence class $[A, B]$ of any pair (A, B). It should be mentioned that the space $\mathcal{K}^2(X)/_\sim$, where $\mathcal{K}(X) = \{A \in \mathcal{B}(X) \mid A \text{ is compact}\}$, plays an important role in quasidifferential calculus.

A pair $(A, B) \in \mathcal{B}^2(X)$ is called *minimal* if there exists no pair $(C, D) \in [A, B]$ with $(C, D) < (A, B)$. For any $(A, B) \in \mathcal{K}^2(X)$ there exists a minimal pair $(A_0, B_0) \in [A, B]$, but this is not true for $\mathcal{B}^2(X)$. There exists a class $[A, B] \in \mathcal{B}^2(c_0)$ which contains no minimal element, where c_0 is the Banach space of all real sequences which converge to zero.

We call a set $A \in \mathcal{B}(X)$ a *summand* of B if there exists $C \in \mathcal{B}(X)$ such that $A \overset{*}{+} C = B$. By $s(A, B)$ we denote the family of all common summands A and B in $\mathcal{B}(X)$.

The pair $(A, B) \in \mathcal{B}^2(X)$ is minimal iff for all $C \in s(A, B)$ the inclusion $C \subset A \overset{*}{+} B$ implies $C = A \overset{*}{+} B$.

A pair $(A, B) \in \mathcal{B}^2(X)$ is called *reduced*, if $A \overset{*}{+} B$ is a summand of C, whenever $C \in s(A, B)$. Clearly, each reduced pair is minimal but not conversely.

The pair $(A, B) \in \mathcal{B}^2(X)$ is reduced iff each element of $[A, B]$ is of the form $(A \overset{*}{+} K, B \overset{*}{+} K)$ with $K \in \mathcal{B}(X)$.

Let $A \vee B$ be the closed convex hull of $A \cup B$. If the pair $(A \vee B, A \overset{*}{+} B)$ is reduced, then (A, B) is reduced.

Triangular Norms, Generalized Quasiconcavity and Compromise Solutions

Milan VLACH and **Jaroslav RAMÍK**

The concept of a triangular norm has an important role in a probabilistic generalization of the theory of metric spaces. From an algebraic point of view, a triangular norm is a semigroup operation on the unit interval of real numbers with identity 1. In the many-valued logic, it serves as a truth function for conjunction.

First we show how the triangular norms lead to a natural generalization of quasiconcavity and strict quasiconcavity of functions mapping a real vector space into the unit interval of real numbers. Then we present elementary properties of these generalizations and discuss their applications to multiobjective optimization.

On Strong Solutions of the Generalized Implicit Vector Variational Problem

Jen-Chih YAO and **Qamrul Hasan ANSARI**

The main motivation of this paper is to prove some existence results for the strong solutions of the generalized implicit vector variational problem which includes generalized vector variational and variational-like inequality problems as special cases. We also establish a fixed point theorem which generalizes known results in the literature.

Harmonic sum and duality

Constantin ZĂLINESCU and Jean-Paul PENOT

We consider an operation on subsets of a topological vector space which is closely related to what has been called the inverse addition by R.T. Rockafellar. Applied to closed convex sets containing zero this operation was considered by A. Seeger and corresponds to the addition under polarity.

However, our study is not limited to the convex case. Crucial tools for it are the gauges one can associate to a subset. We stress the role played by asymptotic cones in such a context. We present an application to the calculus of the conjugate of some functions for one of the most fruitful dualities for quasiconvex problems. We end our work with an extension of the well-known rule for the computation of the normal cone to a convex set defined by a convex inequality.

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