First and second order Characterizations of Generalized Convexity

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Abstract

The purpose of this lecture is to present first and second order characterizations of pseudoconvex and quasiconvex functions.

1 Convex, Quasiconvex and Pseudoconvex Functions

Given C a convex subset of a linear space E and $f : C \to \Re$, f is said to be:

convex on C if for every $x, y \in C$ and $t \in (0,1)$

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)],$$

quasiconvex on C if for every $x, y \in C$ and $t \in (0,1)$

$$f(tx + (1-t)y) \leq max[f(x), f(y)],$$

and *strictly quasiconvex* on C if for every $x, y \in C, x \neq y$ and $t \in (0,1)$

$$f(tx + (1-t)y) < max[f(x), f(y)].$$

The starting point of characterizations of convexity and quasiconvexity of functions of several variables is that they can be expressed in terms of convexity and quasiconvexity of functions of one real variable. Namely, given $a \in C$ and $d \in E$, let us define

$$I_{a,d} = \{t \in \Re : a + td \in C\},$$

and for $t \in I_{a,d}$

$$f_{a,d}(t) = f(a + td).$$

Then, f is convex (quasiconvex) on C if and only if $f_{a,d}$ is convex (quasiconvex) on $I_{a,d}$ for all $a \in C$ and $d \in E$.

Assume that f is differentiable (twice differentiable) on C, then

$$f'_{a,d}(t) = \langle \nabla f(a + td), d \rangle,$$
$$f''_{a,d}(t) = \langle \nabla^2 f(a + td)\, d, d \rangle.$$

$f_{a,d}$ is convex if and only if $f'_{a,d}$ is nondecreasing and also if and only if $f''_{a,d}$ is nonnegative. Hence, first and second order characterizations of convexity for f are straightforwardly derived: f is convex if and only if its gradient ∇f is monotone and also if and only if its Hessian $\nabla^2 f(x)$ is positive semi-definite for all $x \in C$.

In the same spirit, the quest of criteria for quasiconvexity of a function of several variables leads to consider quasiconvexity of functions of one real variable.

Let I be an interval of $\Re$ and $\theta : I \to \Re$. Then θ is quasiconvex on I if and only if there is $t \in \bar{\Re}$ so that (θ is nonincreasing on $(-\infty, t] \cap I$ and nondecreasing on $(t, +\infty) \cap I$) or (θ is nonincreasing on $(-\infty, t) \cap I$ and nondecreasing on $[t, +\infty) \cap I$). When θ is differentiable on I, each of the following conditions:

$$t_1, t_2 \in I \text{ and } \theta(t_1) < \theta(t_2) \quad \Rightarrow \quad \theta'(t_2)(t_2 - t_1) \le 0,$$

$$t_1, t_2 \in I \text{ and } \theta(t_1) \le \theta(t_2) \quad \Rightarrow \quad \theta'(t_2)(t_2 - t_1) \le 0,$$

$$t_1, t_2 \in I \text{ and } \theta'(t_1)(t_2 - t_1) > 0 \quad \Rightarrow \quad \theta'(t_2)(t_2 - t_1) \ge 0,$$

$$\left.\begin{array}{c} t_1, t_3 \in I,\ t_1 < t_2 < t_3, \\ \theta(t_1) < \theta(t_2) \text{ and } \theta'(t_2) = 0 \end{array}\right\} \quad \Rightarrow \quad \theta(t_2) \le \theta(t_3).$$

is a sufficient and necessary condition for θ to be quasiconvex on I. Hence, we straightforwardly derive the following criteria for quasiconvexity of differentiable functions.

Proposition 1.1 *Assume that f is differentiable on the convex set C. Each of the following conditions:*

$$x, y \in C \text{ and } f(y) < f(x) \quad \Rightarrow \quad \langle \nabla f(x), y - x \rangle \leq 0,$$

$$x, y \in C \text{ and } f(y) \leq f(x) \quad \Rightarrow \quad \langle \nabla f(x), y - x \rangle \leq 0,$$

$$x, y \in C \text{ and } \langle \nabla f(x), y - x \rangle > 0 \quad \Rightarrow \quad \langle \nabla f(y), y - x \rangle \geq 0,$$

$$\left.\begin{array}{c} x, x - h, x + th \in C, t > 0 \\ f(x - h) < f(x) \\ \text{and } \langle \nabla f(x), h \rangle = 0 \end{array}\right\} \quad \Rightarrow \quad f(x) \leq f(x + th).$$

is a sufficient and necessary condition for f to be quasiconvex on C.

Unfortunately, simple examples show that f quasiconvex and $\nabla f(x) = 0$ does not necessarily imply that f has a (even local) minimum at x (consider for instance $f(t) = t^3$, $t \in \Re$). In order to remedy this defect, pseudoconvex functions are introduced via a slight modification of the first condition in the proposition above: given a convex set C, a differentiable function $f : C \to \Re$ is said to be *pseudoconvex* [17] on C if

$$x, y \in C \text{ and } f(y) < f(x) \text{ imply } \langle \nabla f(x), y - x \rangle < 0,$$

and *strictly pseudoconvex* on C if

$$x, y \in C,\ x \neq y \text{ and } f(y) \leq f(x) \text{ imply } \langle \nabla f(x), y - x \rangle < 0.$$

A differentiable convex function is pseudoconvex, a differentiable pseudoconvex function is quasiconvex, a differentiable strictly pseudoconvex function is strictly quasiconvex.

Pseudoconvexity of functions of several variables are characterized via pseudoconvexity of functions of one real variable as well. Indeed, f is (strictly) pseudoconvex on C if and only if $f_{a,d}$ is (strictly) pseudoconvex for all $a \in C$ and $d \in E$. Hence, we obtain the following characterizations which are sometimes used as alternative definitions of pseudoconvexity and strict pseudoconvexity.

Proposition 1.2 *Assume that f is differentiable on the convex set C. Then f is pseudoconvex on C if and only if*

$$x, y \in C \text{ and } \langle \nabla f(x), y - x \rangle > 0 \Rightarrow \langle \nabla f(y), y - x \rangle > 0,$$

or equivalently if

$$x, y \in C \text{ and } \langle \nabla f(x), y - x \rangle \geq 0 \Rightarrow \langle \nabla f(y), y - x \rangle \geq 0.$$

Furthermore, f is strictly pseudoconvex on C if and only

$$x, y \in C, x \neq y \text{ and } \langle \nabla f(x), y - x \rangle \geq 0 \Rightarrow \langle \nabla f(y), y - x \rangle > 0.$$

If f is pseudoconvex on the convex set C, $x \in C$ and $\nabla f(x) = 0$, then $f(x) \leq f(y)$ for all $y \in C$ as wished. If C is open, then this property characterizes pseudoconvex functions among quasiconvex functions as seen below.

Theorem 1.1 *([8]) Assume that f is differentiable on the open convex set C.*
(i) If f is pseudoconvex on C then it is quasiconvex on C and has a global minimum at any $x \in C$ such that $\nabla f(x) = 0$,
(ii) If f is quasiconvex on C and has a local minimum at any $x \in C$ such that $\nabla f(x) = 0$, then it is pseudoconvex on C.

The assumption "C is open" is needed: consider the function of two real variables $f(x_1, x_2) = -x_1 x_2$, then f is pseudoconvex on the positive orthant, quasiconvex but not pseudoconvex on the nonnegative orthant.

An immediate consequence is as follows.

Corollary 1.1 *Assume that C is an open convex set C, $f : C \rightarrow \Re$ is differentiable and $\nabla f(x) \neq 0$ for all $x \in C$. Then f is pseudoconvex on C if and only if it is quasiconvex on this set.*

Checking if a twice differentiable function of one real variable is pseudoconvex or quasiconvex leads to consider only the points where the derivative vanishes. For functions of several variables, this consideration leads to introduce the following conditions:

$$(C_1) \qquad x \in C,\ h \in E,\ \langle \nabla f(x), h\rangle = 0 \quad \Rightarrow \quad \langle \nabla^2 f(x)h, h\rangle \geq 0$$

$$(C_2) \qquad \left.\begin{array}{c} h \in E,\ \ x-h, x+th \in C, \\ t>0,\ \ f(x-h) < f(x), \\ 0 = \langle \nabla f(x), h\rangle = \langle \nabla^2 f(x)h, h\rangle \end{array}\right\} \Rightarrow \quad f(x) \leq f(x+th)$$

$$(C_3) \qquad \left.\begin{array}{c} h \in E,\ \ x, x+h \in C, \\ \langle \nabla f(x), h\rangle = \langle \nabla^2 f(x)h, h\rangle = 0 \end{array}\right\} \Rightarrow \quad f(x) \leq f(x+h)$$

$$(\tilde{C}_1) \qquad \left.\begin{array}{c} x \in C,\quad h \in E,\quad h \neq 0, \\ \langle \nabla f(x), h\rangle = 0 \end{array}\right\} \Rightarrow \quad \langle \nabla^2 f(x)h, h\rangle > 0$$

Then, obvious second order characterizations of quasiconvex and pseudoconvex functions of one real variable conduct to the following characterizations for functions of several variables.

Proposition 1.3 *Assume that f is twice differentiable on the convex set C. Then,*
i) f is quasiconvex on C if and only if conditions (C_1) and (C_2) hold,
ii) f is pseudoconvex on C if and only if conditions (C_1) and (C_3) hold.

In the same way, the following sufficient condition for strict pseudoconvexity is easily obtained.

Proposition 1.4 *Assume that C is convex, f is twice differentiable on C and condition $(\tilde{C}_1)$ holds, then f is strictly pseudoconvex on C.*

Thus, if convexity requires positive semi-definiteness of the Hessian, pseudoconvexity and quasiconvexity require also positive semi-definiteness but on the subspace orthogonal to the gradient. Still, Conditions (C_2) or (C_3) in Proposition 1.3 need to be checked at every x where the Hessian is not positive definite on the subspace orthogonal to the gradient. It is not possible to go further with considerations based only on functions of real variable.

The following conditions involve only the points where the gradient vanishes:

$$(C_2') \quad \left.\begin{array}{l} h \in E, \quad x-h, x+th \in C, \\ t>0, \quad f(x-h) < f(x) \\ \nabla f(x) = 0, \ \langle \nabla^2 f(x)h, h\rangle = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \text{there exists } \tilde{t} \in (0,t) \\ \text{so that } f(x) \le f(x+\tilde{t}h) \end{array}\right.$$

$$(C_3') \qquad x \in C, \ \nabla f(x) = 0 \qquad \Rightarrow \ f \text{ has a local minimum at } x$$

The following theorem (Crouzeix [6], see also Crouzeix-Ferland [8]) shows that Conditions (C_2) and (C_3) can be replaced by Conditions (C_2') and (C_3'). The proof, based on the implicit function theorem, involves functions of two real variables.

Theorem 1.2 *Assume that C is an open convex set and f is twice differentiable on C . Then,*
i) f is quasiconvex on C if and only if conditions (C_1) and (C_2') hold,
ii) f is pseudoconvex on C if and only if conditions (C_1) and (C_3') hold.

The following corollary is an immediate consequence of the theorem.

Corollary 1.2 *Assume that C is open and convex, f is twice differentiable on C and $\nabla f(x) \neq 0$ for all $x \in C$. Then,*
f is pseudoconvex on C if and only if condition (C_1) holds.

In the theorem and its corollary, it is assumed that C is open. It is easily seen that if $f : C \to \Re$ is lower semicontinuous on C, $int(C) \neq \emptyset$ and f is quasiconvex on $int(C)$, then f is also quasiconvex on C. This does not work for pseudoconvexity.

Condition (C_1) was early recognized as a necessary condition for quasiconvexity (Arrow-Enthoven [1]). It is easily seen that it is also sufficient for quadratic functions since Conditions (C_2) and (C_3) do not give problems in this case. Katzner [14] proved that, for a twice differentiable function, it is also sufficient when the gradient stays strictly negative on the domain. Theorem 1.2 extends the result of Katzner to more general situations.

2 Convexity and generalized convexity on an affine subspace

In applications, one can often encounter problems of optimization of type

$$\text{minimize } f(x) \text{ subject to } x \in D \cap L$$

where D is a convex set of $\Re^n$ with a nonempty interior and

$$L = \{x : Mx = m\},$$

with $m \in \Re^p$ and M is a $p \times n$ matrix of rank p.

These problems are convex (pseudoconvex, quasiconvex) as soon as the restriction of f to the affine subspace L is convex (pseudoconvex, quasiconvex) on $D \cap L$. Second order conditions are easily obtained.

Theorem 2.1 *Assume that f is twice differentiable on $D \cap L$. Then f is convex on $D \cap L$ if and only if*

$$x \in D \cap L, \quad Mh = 0 \Rightarrow \langle \nabla^2 f(x)h, h\rangle \geq 0.$$

Furthermore, if

$$x \in D \cap L, \quad Mh = 0, \ \ h \neq 0 \Rightarrow \langle \nabla^2 f(x)h, h\rangle > 0.$$

then f is strictly convex on $D \cap L$.

For simplicity, we give only below the transposition of Corrolary 1.2, Transpositions of Theorem 1.2 can also be easily derived, they are left to the reader.

Theorem 2.2 *Assume that f is twice differentiable on $D \cap L$ and $\nabla f(x) \notin M^t(\Re^p)$ for all $x \in int(D) \cap L$.*

Then f is pseudoconvex on $int(D) \cap L$ (and quasiconvex on $D \cap L$) if and only if

$$x \in int(D) \cap L, \ Mh = 0 \ \textit{and} \ \langle \nabla f(x), h\rangle = 0 \Rightarrow \langle \nabla^2 f(x)h, h\rangle \geq 0.$$

Furthermore, if

$$x \in int(D) \cap L, \ Mh = 0, \ h \neq 0 \ \textit{and} \ \langle \nabla f(x), h\rangle = 0 \Rightarrow \langle \nabla^2 f(x)h, h\rangle > 0$$

then f is strictly pseudoconvex on $int(D) \cap L$.

In this theorem, the condition $\nabla f(x) \notin M^t(\Re^p)$ corresponds to the condition $\nabla f(x) \neq 0$ of Corrolary 1.2 .

3 Testing convexity and pseudoconvexity

Condition (C_1) in Theorems 1.2 and 1.3, Theorems 2.1 and 2.2 require to check the positive (semi-)definiteness of the restriction of a quadratic form to a linear subspace. Because this problem appears in many situations: second order optimality conditions, augmented Lagrangean methods, ..., it has been well analysed, see for instance [4], [5], [10], [12] and [13]. A good way to deal with consists to look at the inertia of a bordered matrix associated to the problem. We explain how to do below.

Given a $n \times n$ symmetric matrix H and a $n \times p$ matrix B with rank p, $1 \leq p \leq n-1$, we introduce the bordered matrix:

$$M = \begin{pmatrix} H & B \\ B^t & 0 \end{pmatrix}.$$

M is symmetric, we denote by $In(M) = (\nu_+(M), \nu_-(M), \nu_0(M))$ the inertia of M where $\nu_+(M)$, $\nu_-(M)$, $\nu_0(M)$ are the numbers of positive, negative and null eigenvalues respectively of M. We have $\nu_+(M)+\nu_-(M)+\nu_0(M) = n+p$.

On the other hand, let us consider the conditions:

$$\begin{array}{lccc} (PSD) & B^t x = 0 & \Longrightarrow & \langle x, Hx \rangle \geq 0 \\ (PD) & x \neq 0,\ B^t x = 0 & \Longrightarrow & \langle x, Hx \rangle > 0 \end{array}$$

The following theorem relates the inertia of M to conditions (PSD) and (PD).

Theorem 3.1

1. *$\nu_+(M) \geq p$ and $\nu_-(M) \geq p$,*
2. *if (PSD) holds, then H has at most p negative eigenvalues,*
3. *(PD) holds if and only if there is $r > 0$ such that $H + rBB^t$ is positive definite,*
4. *(PSD) holds if and only if $\nu_-(M) = p$,*
5. *(PD) holds if and only if $\nu_+(M) = n$.*

This theorem needs to compute the inertia of a symmetric matrix, this can be efficiently done using Shur's complements (see for instance Cottle [5]). Let us consider the partitionned matrix

$$M = \begin{pmatrix} P & Q \\ Q^t & R \end{pmatrix}$$

with P and R symmetric, P nonsingular. Then

$$In(M) = In(P) + In(R - Q^t P^{-1} Q).$$

$R - Q^t P^{-1} Q$ is called the Shur's complement of M by P. When specialized to the particular case where B is a column matrix, i.e., $B = b$ with $b \in \Re^n$, $b \neq 0$, Theorem 3.1 becomes:

Theorem 3.2

1. *$\nu_+(M) \geq 1$ and $\nu_-(M) \geq 1$,*

2. *if (PSD) holds, then H has at most one negative eigenvalue,*

3. *(PD) holds if and only if there is $r > 0$ such that $H + rbb^t$ is positive definite,*

4. *(PSD) holds if and only if one of the two following conditions hold:*

   (a) $\nu_-(H) = 0$, i.e., H is positive semi-definite,

   (b) $\nu_-(H) = 1$, $b \in H(\Re^n)$ and $\langle H^\dagger b, b\rangle \leq 0$.

5. *(PD) holds if and only if one of the three following conditions hold:*

   (a) $\nu_+(H) = n$, i.e., H is positive definite,

   (b) $\nu_+(H) = n - 1$, $\nu_-(H) = 0$ and $b \notin H(\Re^n)$,

   (c) $\nu_-(H) = 1$, $b \in H(\Re^n)$ and $\langle H^\dagger b, b\rangle < 0$.

In this theorem, $H^\dagger$ denotes the pseudoinverse of H. To get Theorem 3.2 from Theorem 3.1, consider matrices P and D with D diagonal, $P^t P = I$, $H = P^t D P$. Take also $c \in \Re^n$ so that $b = P^t c$. Then use the Lagrange-Sylvester law on inertia and the result on Shur's complements.

It straightforwardly follows that if f is twice differentiable and quasiconvex on an open convex set C, then its Hessian has at most one negative eigenvalue. If, in addition, f is additively separable, i.e., if f has the following form

$$f(x) = \sum_{i=1}^{i=p} f_i(x_i) \quad \text{for } x = (x_1, x_2, \cdots, x_p) \in C = C_1 \times C_2 \times \cdots, C_p,$$

where, for all i, $C_i \subset \Re^{n_i}$ is open and convex and f_i is not constant on C_i, then all f_i are convex except perhaps one.

4 A few examples

4.1 Quasiconvex quadratic functions

Quasiconvex quadratic functions have been investigated by Martos [18], Ferland [11] and Schaible [19]. We show how the above characterizations apply. Let us consider the function:

$$f(x) = \frac{1}{2}\langle Ax, x\rangle - \langle a, x\rangle + \alpha$$

where A is a $n \times n$ symmetric matrix, $a \in \Re^n$ and $\alpha \in \Re$. Let C be a convex set with nonempty interior. Thanks to Theorems 1.2 and 3.2, we see that f is quasiconvex on C (and pseudoconvex on $int(C)$) if and only if one of the following conditions holds:

1. A is positive semi-definite, or

2. $\nu_-(A) = 1$ and for all $x \in int(C)$ one has $Ax - a \in A(\Re^n)$ and $\langle A^\dagger(Ax - a), (Ax - a)\rangle \leq 0$.

In the first case f is convex on the whole space $\Re^n$. In the second case $\bar{x}$ exists so that $A\bar{x} = a$ and $\langle (x - \bar{x}), A(x - \bar{x})\rangle \leq 0$ for all $x \in int(C)$. It can be shown that

$$\{u \ : \ \langle Au, u\rangle \leq 0\} \ = \ T \cap -T,$$

where T is a closed convex cone with nonempty interior. To summarize, we have

Proposition 4.1 *f is quasiconvex on C (and pseudoconvex on $int(C)$) if and only if one of the following conditions holds:*

1. *A is positive semi-definite,*

2. *$\nu_-(A) = 1$, there is $\bar{x}$ so that $A\bar{x} = a$ and either $C \subset \bar{x}+T$ or $C \subset \bar{x}-T$.*

4.2 Cobb-Douglas functions

Cobb-Douglas functions are defined on the positive orthant of $\Re^n$ by

$$f(x) = \prod x_i^{a_i}$$

with $a_i \neq 0$ for $i = 1, 2, \cdots, n$. We are interested in the concavity or pseudo-concavity of f on the orthant. We have

$$\frac{1}{f(x)}\nabla f(x) = X^{-1}Ae, \quad \frac{1}{f(x)}\nabla^2 f(x) = X^{-1}A(ee^t - A^{-1})AX^{-1},$$

where $X = diag(x)$, $A = diag(a)$, and $e = (1, 1, \cdots, 1)^t$. In view of the Lagrange-Sylvester's law on inertia, we have to consider the inertia of the matrices

$$P = A^{-1} - ee^t \quad \text{and} \quad Q = \begin{pmatrix} A^{-1} - ee^t & e \\ e^t & 0 \end{pmatrix},$$

respectively. Indeed, f is concave if $\nu_-(P) = 0$ and pseudoconcave if $\nu_-(Q) = 1$.

To compute these inertia, we consider the matrices:

$$M = \begin{pmatrix} 1 & e \\ e^t & A^{-1} \end{pmatrix}, \quad \text{and} \quad N = \begin{pmatrix} 1 & e & 0 \\ e^t & A^{-1} & e \\ 0 & e^t & 0 \end{pmatrix}.$$

Using Shur's complements, we have on the first hand

$$In(M) = (1, 0, 0) + In(P) \quad \text{and} \quad In(N) = (1, 0, 0) + In(Q).$$

and on the second hand

$$\begin{aligned} In(M) &= In(A^{-1}) + In(1 - \langle a, e \rangle), \\ In(N) &= In(A^{-1}) + In(-\langle a, e \rangle) + In(1). \end{aligned}$$

It follows that f is concave on the positive orthant if and only if all a_is are positive and $\sum a_i \leq 1$ and quasi concave on the positive orthant if and only if either all a_is are positive or all a_is are positive except one and $\sum a_i \leq 0$.

Convexity and pseudoconvexity of f can be treated in the same way.

5 Selected references

A lot of papers have been published on these questions. For the help of the readers we have retained only a few among them.

In the present paper no proofs are given. They can be found in [7] which contains also historical comments and a more complete bibliography. First order characterizations of generalized monotonicity of maps are also considered in this reference.

References [3, 20] are two textbooks on generalized convexity.

References [11, 16, 18] are concerned mainly with quadratic generalized convex functions.

References [4, 5, 10, 13, 12] are concerned with positive (semi-)definiteness of a quadratic form on a linear subspace.

Other references are concerned with second order conditions for generalized convexity.

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