Continuity and Differentiability of Quasiconvex Functions

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Abstract

Continuity and differentiability of quasiconvex functions are studied. The connection with the monotonicity of real functions of several variables receives a special emphasis.

1 Introduction and notation

The properties of convex functions are well known. A convex function of one real variable admits right and left derivatives at any point in the interior of its domain, hence it is continuous at such a point. A convex function f defined on a normed linear space E is continuous at $x \in E$ if bounded in a neighbourhood of x. If, in addition, $E = \mathbb{R}^n$ and x belongs to the interior of the domain of f, then f is continuous at x, the directional derivatives $f'(x,d)$ of f at x with respect to the directions d are well defined. Furthermore, $f'(x,d)$ is convex in d, and the subgradient $\partial f(x)$ of f at x is defined as the closed convex set such that

$$\partial f(x) = \{\, x^\star \,:\, \langle d, x^\star \rangle \le f'(x,d) \text{ for all } d \,\}.$$

It is important to notice that all these properties are due to the geometrical structures induced by the convexity of f. Indeed, the strict epigraph of f is convex in $E \times \mathbb{R}$. Let $x \in int(dom f)$, then

$$\{(x, f(x))\} \cap \{(y, \lambda) \, : \, f(y) < \lambda\} \; = \; \emptyset.$$

Let us define the following set

$$T(a) = \{(x^\star, \lambda^\star) : \langle x^\star, y - x \rangle + \lambda^\star(\lambda - f(x)) \leq 0 \; \forall \, (y, \lambda) \text{ s.t. } f(y) < \lambda\} \quad (1)$$

Then, in view of separation theorems on convex sets, $T(a)$ is a nonempty closed convex cone and $\partial f(a)$ is obtained from $T(a)$ as follows

$$x^\star \in \partial f(a) \Longleftrightarrow (x^\star, -1) \in T(a).$$

The epigraph of a quasiconvex function is not convex, hence separation theorems do not apply, thence it is not possible to deal in a similar way. Still, convexity is present through the lower level sets of the function and it gives birth to some interesting properties. The purpose of this lecture is to show how to exploit this convexity.

Through the lecture, we use the following notation:

Let E be a normed linear real space and $f : E \to [-\infty, +\infty]$ (as usual in convex analysis, we consider functions defined on the whole space; if it is not the case set $f(x) = +\infty$ for x not in the domain).

For $\lambda \in (-\infty, -\infty)$ let us define

$$\begin{array}{rll} & S_\lambda(f) & = \{x \, : \, f(x) \leq \lambda\}, \\ \text{and} & \tilde{S}_\lambda(f) & = \{x \, : \, f(x) < \lambda\}. \end{array}$$

Clearly, for $\lambda < \mu$,

$$\tilde{S}_\lambda(f) \subseteq S_\lambda(f) \subseteq \tilde{S}_\mu(f) \subseteq S_\mu(f).$$

It is also easily seen that

$$S_\lambda(f) = \cap_{\mu > \lambda} \tilde{S}_\mu(f) = \cap_{\mu > \lambda} S_\mu(f).$$

The function f can be recovered from its level sets, indeed:

$$f(x) = \inf \left[\, \lambda : x \in S_\lambda(f) \, \right] = \inf \left[\, \lambda : x \in \tilde{S}_\lambda(f) \, \right].$$

By definition, f is said to be *quasiconvex* on E if

$$x, y \in E,\ 0 < t < 1 \Longrightarrow f(tx + (1-t)y) \leq max[\, f(x), f(y)\,],$$

and *strictly quasiconvex* on $C \subseteq E$, C being convex, if

$$x, y \in C,\ x \neq y,\ 0 < t < 1 \Longrightarrow f(tx + (1-t)y) < max[\, f(x), f(y)\,].$$

Quasiconvexity has a geometrical interpretation, indeed

$$f \text{ quasiconvex } \Leftrightarrow S_\lambda(f) \text{ convex } \forall \lambda \in \mathbb{R} \Leftrightarrow \tilde{S}_\lambda(f) \text{ convex } \forall \lambda \in \mathbb{R}.$$

Recall that for a function $f : E \to [-\infty, +\infty]$,

$$f \text{ lower semi-continuous (lsc in short)} \Leftrightarrow S_\lambda(f) \text{ closed } \forall \lambda \in \mathbb{R}$$

and

$$f \text{ upper semi-continuous (usc in short)} \Leftrightarrow \tilde{S}_\lambda(f) \text{ open } \forall \lambda \in \mathbb{R}.$$

A set $C \subseteq E$ is said to be *evenly convex* if it is the intersection of open half spaces. It results from separation theorems that open and closed convex sets are evenly convex.

A function f is said to be *evenly quasiconvex* if all $S_\lambda(f)$ are convex. Lower semi-continuous quasiconvex functions and upper semi-continuous quasiconvex functions are evenly quasiconvex.

Sometimes, infinite values are not well praised when considering continuity and differentiability. It is easy to avoid this problem:

Given $g : E \to [-\infty, +\infty]$, let us consider

$$f(x) = \arctan\, g(x).$$

Then $f : E \to [-\frac{\pi}{2}, +\frac{\pi}{2}]$. It is easy to see that f is (evenly) quasiconvex if and only if g is (evenly) quasiconvex and f is lsc (usc) at a point a if and only if g is so. In this spirit, we shall say that g is differentiable at a if f is differentiable at this point. Thus, for continuity and differentiability questions in quasiconvex analysis, it is sufficient to consider functions which are finite on the whole space.

2 Regularized quasi convex functions

Given a family $\{T_\lambda\}_{\lambda\in\mathbb{R}}$ such that

$$\emptyset \subseteq T_\lambda \subseteq T_\mu \subseteq E \text{ for all } \lambda,\mu \in \mathbb{R} \text{ such that } \lambda < \mu,$$

we define a function $g : E \to [-\infty, +\infty]$ by

$$g(x) \;=\; \inf\,[\,\lambda \,:\, x \in T_\lambda].$$

Then,

$$S_\lambda(g) \;=\; \cap_{\mu>\lambda} T_\mu.$$

Given $f : E \to [-\infty, +\infty]$, we apply this process to the sets $T_\lambda = cl(S_\lambda(f))$, $T_\lambda = co(S_\lambda(f))$, $T_\lambda = eco(S_\lambda(f))$, and $T_\lambda = \overline{co}(S_\lambda(f))$ where $cl(S)$, $co(S)$, $eco(S)$ and $\overline{co}(S)$ denote the closure (i.e. the closed hull), the convex hull, the evenly convex hull and the closed convex hull of S respectively. We denote by $\bar{f}$, f_q, f_e and $f_{\bar{q}}$ respectively the different functions obtained with the process. Because any intersection of closed (convex, evenly convex, closed convex) sets is closed (convex, evenly convex, closed convex) , then $\bar{f}$, f_q, f_e and $f_{\bar{q}}$ are respectively the greatest lower semi-continuous, quasiconvex, evenly quasiconvex, and lsc quasiconvex functions bounded from above by f. If is seen that $\bar{f} = f_{\bar{q}}$ when f is quasiconvex.

The proof of the following result is rather easy and left to the reader.

Proposition 2.1 *Let $f : E \to [-\infty, +\infty]$. Then*

$$f \text{ is lsc at } a \Longleftrightarrow \bar{f}(a) = f(a).$$

On the other hand, since

$$S_\lambda(\bar{f}) \;=\; \cap_{\mu>\lambda} cl(S_\mu(f)) \text{ and } S_\lambda(f) \;=\; \cap_{\mu>\lambda} S_\mu(f),$$

it follows that for all $\lambda \in \mathbb{R}$

$$cl(S_\lambda(f)) \subseteq S_\lambda(\bar{f}).$$

The equality does not hold for a general function. However, for a quasiconvex function, we have the following result:

Proposition 2.2 *Assume that f is quasiconvex and $int(S_\lambda(f)) \neq \emptyset$. Then*

$$cl(S_\lambda(f)) = S_\lambda(\bar{f}).$$

Proof: Fix $a \in int(S_\lambda(f))$. Let $x \in S_\lambda(\bar{f})$, then $x \in cl(S_\mu(f))$ for all $\mu > \lambda$ and $x_t = x + t(a - x) \in int(S_\mu(f))$ for all $t \in (0, 1)$. It follows that $x_t \in S_\lambda(f)$ and therefore $x \in cl(S_\lambda(f))$. ■

Furthermore, for quasiconvex functions, we have also the following result:

Proposition 2.3 *Assume that $f : \mathbb{R}^n \to [-\infty, +\infty]$ is quasiconvex. Then f is continuous at $\bar{x}$ if and only if $\bar{f}$ is continuous at $\bar{x}$.*

Proof: Assume that f is continuous at $\bar{x}$. Then $\bar{f}$ is lsc by definition, it is also usc at $\bar{x}$ since $f_{\bar{q}}(\bar{x}) = f(\bar{x})$, $f_{\bar{q}} \leq f$ and f is usc at $\bar{x}$. Next, assume that $\bar{f}$ is continuous at $\bar{x}$. Let $\lambda > f(\bar{x})$. Then $S_\lambda(\bar{f}) = cl(S_\lambda(f))$ is a neighbourhood of $\bar{x}$. Since $int(cl(S)) = int(S)$ for a convex set S with $int(S) \neq \emptyset$, then $S_\lambda(f)$ is a neighbourhood of $\bar{x}$ as well. The conclusion follows. ■

Evenly convex sets have been introduced by Fenchel [10]. Evenly quasiconvex functions have been introduced by Passy and Prisman [13] and, independently, by Martinez-Legaz [11]. The description of the process for constructing regularized functions is given in Crouzeix [3, 6].

3 Quasiconvexity and monotonicity

Let $\theta : \mathbb{R} \to [-\infty, +\infty]$. Then θ is quasiconvex if and only if there exists $t \in [-\infty, +\infty]$ so that:

- either θ is nonincreasing on $(-\infty, t]$ and nondecreasing on $(t, +\infty) \cap I$,
- or θ is nonincreasing on $(-\infty, t)$ and nondecreasing on $[t, +\infty) \cap I$.

Thus, the simplest examples of quasiconvex functions are the nondecreasing functions of one real variable. It results that, unlike convex functions, quasiconvex functions are not continuous in the interior of their domain. A fortiori, directional derivatives are not necessarily defined. Still, nondecreasing functions of one real variable are almost everywhere continuous and differentiable, hence quasiconvex functions of one real variable are also almost everywhere continuous and differentiable on the interior of their domains.

Quasiconvex functions of several variables have also connections with monotonicity. Let E be a normed linear space, $f : E \to [-\infty, +\infty]$ and K be a convex cone of E, f is said to be *nondecreasing with respect to* K if

$$x, y \in E, \;\; y - x \in K \Longrightarrow f(x) \leq f(y).$$

Theorem 3.1 *Let $f : E \to (-\infty, +\infty)$, f quasiconvex, $\lambda \in \mathbb{R}$ and $a \in E$ such that $int(S_\lambda(f)) \neq \emptyset$ and $a \notin cl(S_\lambda(f))$. Then there exists an open convex neighbourhood V of a and a nonempty open convex cone K so that*

$$x, y \in V, \; y - x \in K \Longrightarrow f(x) \leq f(y).$$

Futhermore, if f is strictly quasiconvex

$$x, y \in V, \; y - x \in K, \; x \neq y \Longrightarrow f(x) < f(y).$$

Proof: Let $b \in int(S_\lambda(f))$, $r > 0$ and $R > 0$ be such that $B(b, r) \subseteq S_\lambda(f)$ and $S_\lambda(f) \cap B(a, R) = \emptyset$. Let some $\bar{t} > 0$. Set $c = a + \bar{t}(a - b)$ and

$$K \;=\; \{d \,:\, c - td \in B(b, r) \text{ for some } t > 0\}.$$

Then K is a nonempty open convex cone. Hence $y - K \subseteq c - K$ for all $y \in c - K$. Set $V = (c - K) \cap B(a, R)$. Assume that $x, y \in V$ with $y - x \in K$. Then there is $t > 1$ so that $z = y + t(x - y) \in B(b, r)$. Notice that $f(z) \leq \lambda < f(y)$. The results follows from the (strict) quasiconvexity of f. ∎

In the particular case where $E = \mathbb{R}^n$, we derive the following result.

Corollary 3.1 *Let $f : \mathbb{R}^n \to (-\infty, +\infty)$, f quasiconvex, $\lambda \in \mathbb{R}$ and $a \in \mathbb{R}^n$ such that $int(S_\lambda(f)) \neq \emptyset$ and $a \notin cl(S_\lambda(f))$. Then there exist an open convex neighbourhood V of a and $v_1, v_2, \cdots, v_n$, n linearly independent vectors such that*

$$x, x + \sum t_i v_i \in V, \; t_1, t_2, \cdots, t_n \geq 0 \Longrightarrow f(x) \leq f(x + \sum t_i v_i).$$

Futhermore, if f is strictly quasiconvex and $\sum t_i > 0$, then the inequality is strict.

Proof: Choose for vectors v_i n linearly independent vectors in K. ■

In $\mathbb{R}^n$, locally Lipschitz functions also are strongly connected to monotonicity. Indeed, assume that f is locally Lipschitz in a neighbourhhod of a, i.e., there is $r > 0$ and $L > 0$ such that

$$x_i, y_i \in [a_i - r, a_i + r] \text{ for all } i \Longrightarrow |f(y) - f(x)| \leq L \sum |y_i - x_i|.$$

Define $g(x) = f(x) + L \sum x_i$. Then $a_i - r \leq x_i \leq y_i \leq a_i + r$ for all i implies $g(x) \leq g(y)$.

4 Continuity

Given $f : E \to [-\infty, +\infty]$, $a \in E$ with $-\infty < f(a) < +\infty$ and a direction $d \in E$, we define the function of one real variable

$$f_{a,d}(t) = f(a + td).$$

The first result concerns nondecreasing functions.

Proposition 4.1 *Assume that $f(a)$ is finite, K is a convex cone with nonempty interior, f is nondecreasing with respect to K and $d \in int(K)$. Then f is lsc (usc) at a if and only if $f_{a,d}$ is lsc (usc) at* 0.

Proof: Assume that $f_{a,d}$ is lsc (usc) at 0. Let $\lambda_- < f(a)$ ($\lambda_+ > f(a)$). Then there is $t_- < 0$ ($t_+ > 0$) such that $\lambda_- < f(a+td)$ for all $t \geq t_-$ ($\lambda_+ > f(a+td)$ for all $t \leq t_+$). Take $V = (a + t_-d) + K$ ($V = (a + t_+d) - K$). V is a neighbourhood of a and $\lambda_- < f(a + t_-d) \leq f(x)$ ($\lambda_+ > f(a + t_+d) \geq f(x)$) for all $x \in V$. ■

We have seen (Theorem 3.1) that quasiconvex functions can be considered locally nondecreasing with respect to some open convex cone K. Hence the last proposition can be applied. However, a stronger result holds. Assume that f is quasiconvex and $f(a)$ is finite. Let us define

$$\tilde{K}(a) = \{d : f(a + td) < f(a) \text{ for some } t > 0\} \tag{2}$$

$\tilde{K}(a)$ is a convex cone. Its interior is nonempty as soon as $int(\tilde{S}_{f(a)}(f)) \neq \emptyset$. Notice that $\tilde{K}(a)$ contains the cones K of Theorem 3.1. Then, we have:

Proposition 4.2 *Assume that f is quasiconvex, $f(a)$ is finite and $d \in int(\tilde{K}(a))$. Then f is lsc (usc) at a if and only if $f_{a,d}$ is lsc (usc) at 0.*

Proof: Combine the proof of the last proposition with the proof of Theorem 3.1. ■

In these two propositions, we have considered continuity in one direction d belonging to a specific cone. The following result can appear weaker since it involves continuity in all directions, but it deserves to be given in reason of its very simple formulation.

Theorem 4.1 *[3, 6] Assume that f is quasiconvex on $\mathbb{R}^n$ and $f(a)$ is finite. Then f is lsc (usc) at a if and only if, for all $d \in \mathbb{R}^n$, the function $f_{a,d}$ is lsc (usc) at 0.*

Proof: Assume that, for all d, $f_{a,d}$ is lsc at 0 and prove that f is lsc at a. If $\tilde{S}_{f(a)}(f) = \emptyset$ there is nothing to prove. If not take some d in the relative interior of $\tilde{K}(a)$ and adapt the proof of the last proposition.

Next, assume that, for all d, $f_{a,d}$ is usc at 0 . Let $\lambda > f(a)$. Take $d = e_i$ be the i-th vector of the canonical basis of $\mathbb{R}^n$. There is $t_i > 0$ such that $f(a + td_i) < \lambda$ for all $t \in [-t_i, t_i]$. Take for V the convex hull of the $2n$ points $a \pm t_i d_i$. Then V is a neigborhood of a and $V \subseteq S_\lambda(f)$. ■

This result says that, in $\mathbb{R}^n$, a quasiconvex function is continuous at a if it is continuous along the lines at this point. Such a result does not hold for nondecreasing functions as it can be seen from easily obtained examples. Also, it does not hold for an infinite dimensional linear space: take a linear but not continuous function.

5 Differentiability: notation and first results

We start with the notation. Assume that $f(a)$ is finite and $h \in E$. Then the *upper* and the *lower Dini-derivative of f at a with respect to the direction h* are respectively defined by

$$\begin{aligned} f'_+(a,h) &= \limsup_{t \to 0_+} \frac{f(a+th) - f(a)}{t}, \\ f'_-(a,h) &= \liminf_{t \to 0_+} \frac{f(a+th) - f(a)}{t}. \end{aligned}$$

If $-\infty < f'_-(a,h) = f'_+(a,h) < \infty$ then *the directional derivative of f with respect to the direction h* exists and is defined by

$$f'(a,h) = f'_-(a,h) = f'_-(a,h).$$

If for all $h \in E$

$$f'(a,h) \text{ and } f'(a,-h) \text{ exist and } f'(a,h) + f'(a,-h) = 0$$

then f is said to be *differentiable at a along the lines* or again *weakly Gateaux-differentiable* at a.

If there is a vector $c \in E'$ such that

$$f'(a,h) = \langle c, h\rangle \qquad \forall h \in E$$

then f is said to be *Gateaux-differentiable* at a. Such a c is uniquely defined. It is called the *(Gateaux-) gradient* of f at a and denoted by $\nabla f(a)$.

If f is Gateaux-differentiable at a and

$$\frac{f(a+h) - f(a) - \langle \nabla f(a), h\rangle}{\|h\|} \to 0 \text{ when } h \to 0$$

then f is said to be *Fréchet-differentiable* at a.

Although Fréchet- and Gateaux-differentiability do not coincide for a general function, they coincide when the function is monotone on $\mathbb{R}^n$.

Theorem 5.1 *(Chabrillac-Crouzeix [2]) Let $f : \mathbb{R}^n \to [-\infty, \infty]$, K be a nonempty open convex cone. Assume that f is nondecreasing with respect to K and f is Gateaux-differentiable at a. Then f is Fréchet-differentiable at a.*

Proof: Assume that f is Gateaux- but not Fréchet-differentiable at a. Then there exist $\epsilon > 0$ and a sequence $\{h_n\}_n$ converging to 0 such that for all n

$$7\epsilon < \frac{|f(a+h_n) - f(a) - \langle \nabla f(a), h_n\rangle|}{\|h_n\|}. \tag{3}$$

Set $d_n = \frac{1}{\|h_n\|} h_n$. Without loss of generality, we can assume that all the sequence $\{d_n\}_n$ converges to some $\bar{d}$. Let $e \in int(K)$. Then $\mu > 0$ exists so that

$$|\langle \nabla f(a), d - \bar{d}\rangle| < \epsilon \text{ for all } d \in V \tag{4}$$

where

$$V = \{d : \bar{d} - \mu e = d_- \le d \le d_+ = \bar{d} + \mu e\}.$$

Then V is a neighbourhood of $\bar{d}$. For n large enough, $d_n \in V$ and therefore

$$f(a + \|h_n\| d_-) - f(a) \le f(a + h_n) - f(a) \le f(a + \|h_n\| d_+) - f(a). \quad (5)$$

Since f is Gateaux-differentiable at a, for n large enough

$$\frac{|f(a + \|h_n\| d_-) - f(a) - \langle \nabla f(a), \|h_n\| d_- \rangle|}{\|h_n\|} < \epsilon, \quad (6)$$

and

$$\frac{|f(a + \|h_n\| d_+) - f(a) - \langle \nabla f(a), \|h_n\| d_+ \rangle|}{\|h_n\|} < \epsilon. \quad (7)$$

The contradiction follows from Equations (3), (4), (5), (6) and (7). ■

As an immediate corollary, we see that Gateaux- and Fréchet-differentiability coincide for locally Lipschitz functions on $\mathbb{R}^n$. Also, Theorem 5.1 can be applied to quasiconvex functions.

Theorem 5.2 *Assume that $f : \mathbb{R}^n \to [-\infty, \infty]$ is quasiconvex. If f is Gateaux-differentiable at a, then f is Fréchet-differentiable at a as well.*

Proof: Let $S = \{x : f(x) < f(a)\}$.

If $int(S) \neq \emptyset$, there is $\lambda \in \mathbb{R}$ such that $int(S_\lambda(f)) \neq \emptyset$ and $a \notin cl(S_\lambda(f))$. The result follows from Theorems 3.1 and 5.1.

If $S = \emptyset$. Then $f(x) \ge f(a)$ for all x and therefore $\nabla f(a) = 0$. Let $(e_1, e_2, \cdots, e_n)$ be the canonical basis of $\mathbb{R}^n$. Set $e_{i+n} = -e_i$ for $i = 1, \cdots, n$. For $h \in \mathbb{R}^n$ take $t_i = max[0, h_i]$ and $t_{i+n} = max[0, -h_i]$. Then

$$a + h = a + \sum_{i=1}^{i=2n} t_i e_i = \sum_{i=1}^{i=2n} (a + \frac{t_i}{\|h\|} \|h\| e_i)$$

where $\|h\| = \sum_{i=1}^{i=n} |h_i| = \sum_{i=1}^{i=2n} t_i$. Then, since f is quasiconvex

$$0 \le \frac{f(a + th) - f(a)}{\|h\|} \le \max_{i=1,\cdots,2n} \frac{f(a + \|h\| e_i) - f(a)}{\|h\|},$$

and the result follows again.

We are left with the case where $int(S) = \emptyset$ but $S \neq \emptyset$. Here again $\nabla f(a) = 0$. The proof is obtained in working on the affine set generated by S and using the same proof as above. ∎

A previous proof by Crouzeix [8] of that theorem does not make use of Theorem 5.1.

It is well known that a nondecreasing function of one real variable is almost everywhere differentiable. The result still holds in $\mathbb{R}^n$.

Theorem 5.3 *(Chabrillac-Crouzeix [2]) Let $f : \mathbb{R}^n \to [-\infty, \infty]$, K be a nonempty open convex cone. Assume that f is nondecreasing with respect to K. Then f is almost everywhere Fréchet-differentiable.*

The celebrated Rademacher's theorem on locally Lipschitz functions can be viewed as a corollary of this theorem. Another consequence is that quasiconvex functions on $\mathbb{R}^n$ are also almost everywhere Fréchet-differentiable, a previous and direct proof of this result was given in Crouzeix [7].

6 Directional derivatives

It is clear that, for a general function, the Dini-derivatives $f'_-(a, h)$ and $f'_+(a, h)$ are positively homogeneous of degree 1 with respect to the direction h. This is also the case for the directional derivative $f'(a, h)$ when defined. The following proposition is a direct consequence of quasiconvexity.

Proposition 6.1 *Assume that $f : E \to [-\infty, \infty]$, f is quasiconvex and $f(a)$ is finite. Then $h :\to f'_+(a, h)$ is quasiconvex.*

There are counter-examples (Crouzeix [5]) where the function f is quasiconvex, but the lower Dini-derivative is not. The fact that, for a convex function, the directional derivative $f'(a, h)$ is convex and positively homogeneous in h has strong implications. Indeed, the indicator function of $\partial f(a)$, the Fenchel-subgradient of f at a, is nothing else that the Fenchel-conjugate of the function $f'(a, .)$. Quasiconvex positively homogeneous functions also will play a fundamental role as seen below.

Theorem 6.1 *(Newman [12], Crouzeix [4]) Assume that $C \subseteq E$ is convex and $\theta : C \to (-\infty, +\infty]$ is quasiconvex and positively homogeneous of degree one.*

1. *If $\theta(x) < 0$ for all $x \in C$, then θ is convex on C,*

2. *If $f\,\theta(x) \geq 0$ for all $x \in C$, then θ is convex on C.*

Proof: Several types of proofs can be given. The following one is based on the geometrical aspect of convexity. Consider in case 1)

$$S \;=\; S_{-1}(\theta) \times \{-1\} \subseteq E \times \mathbb{R},$$

and in case 2)

$$S \;=\; S_1(\theta) \times \{1\} \subseteq E \times \mathbb{R}.$$

Next, let us consider the consider the cone T in $E \times \mathbb{R}$ generated by S. Both S and T are convex. In case 1) T corresponds to the epigraph of θ. In case 2) take $\tilde{T} = T \cup (S_\infty \times \{0\})$ where S_∞ is the recession cone of $S_1(\theta)$. Then, $\tilde{T}$ corresponds to the the epigraph of θ as well. ∎

Now, assume that $\theta : \mathbb{R}^n \to (-\infty, +\infty]$ is quasiconvex and positively homogeneous of degree 1. Define θ_- and θ_+ by

$$\begin{array}{llllll} \text{if} & \theta(x) < 0 & \text{then} & \theta_-(x) = \theta(x) & \text{and} & \theta_+(x) = 0, \\ \text{if} & \theta(x) \geq 0 & \text{then} & \theta_-(x) = +\infty & \text{and} & \theta_+(x) = \theta(x) \end{array} \tag{8}$$

Then $\theta(x) = \min[\,\theta_-(x), \theta_+(x)\,]$. Hence θ is the minimum of two convex functions. This decomposition will be useful for the directional derivatives and the upper Dini-derivatives of quasiconvex functions, indeed they are quasiconvex in the direction.

Theorem 6.2 *(Crouzeix [8]) Assume that $f : E \to [-\infty, \infty]$ is quasiconvex, $f(a)$ is finite and f is weakly Gateaux-differentiable at a. Then f is Gateaux-differentiable at a as well.*

Proof: By assumption, $f'(a,h) + f'(a,-h) = 0$ for all h. If $f'(a,h) = 0$ for all h, take $\nabla f(a) = 0$. Next, assume that $f'(a,h) \neq 0$ for some h. Set $\theta(h) = f'(a,h)$, $S_- = \{h \in E \,:\, f'(a,h) < 0\}$, $S_0 = \{h \in E \,:\, f'(a,h) \leq 0\}$ and $S_+ = \{h \in E \,:\, f'(a,h) > 0\}$. Then, $S_+ = -S_-$. The sets S_-, S_0 and S_+ are convex. S_0 and S_+ are two complementarity sets. Hence the closure of S_0 is an half space. The functions θ_- and θ_+ are convex on S_- and S_+ respectively, furthermore $\theta_-(h) = -\theta_+(-h)$ for all $h \in S_- = -S_+$. Hence θ_- is linear on S_-. Finally, it is seen that θ is linear on the whole space. ∎

It follows that, for quasiconvex functions on $\mathbb{R}^n$, Fréchet-, Gateaux- and weakly Gateaux-differentiability coincide. For nondecreasing functions on $\mathbb{R}^n$, Gateaux- and weakly Gateaux-differentiability do not coincide in general.

The next theorem is to be compared with Proposition 4.2. The proof being not immediate, the reader is directed to Crouzeix [5]. As in Proposition 4.2, it involves the convex cone

$$\tilde{K}(a) = \{d \,:\, f(a+td) < f(a) \text{ for some } t > 0\}.$$

If f is differentiable at a and $\nabla f(a) \neq 0$, then the closure of $\tilde{K}(a)$ is an half-space. The theorem corresponds to a converse implication.

Theorem 6.3 *Assume that f is quasiconvex on $\Re^n$, $f(a)$ is finite, the closure of $\tilde{K}(a)$ is an half space and $d \in int(\tilde{K}(a))$. Then f is Gateaux-differentiable at a if and only if the function of one real variable $f_{a,d}$ is differentiable at* 0.

7 More on the Dini-derivatives

The first result concerns nondecreasing functions. The proof is easy and left to the reader.

Proposition 7.1 *Assume that K is a convex cone with $int(K) \neq \emptyset$ and f is nondecreasing with respect to K. Then $f'_-(a,.)$ and $f'_+(a,.)$ are nondecreasing with respect to K. Hence they are continuous on $K \cup -K$.*

If f is quasiconvex, the result still holds with $\tilde{K}(a)$ instead of K.

Proposition 7.2 *Assume f quasiconvex and $f(a)$ is finite. Then*

- *$f'_-(a,.)$ and $f'_+(a,.)$ are nondecreasing with respect to $\tilde{K}(a)$,*
- *$f'_-(a,.)$ and $f'_+(a,.)$ are continuous on $int(\tilde{K}(a))$ and $-int(\tilde{K}(a))$.*

The following result [5] makes more clear the connection between the Dini-derivatives and the cone $\tilde{K}(a)$.

Proposition 7.3 *Assume f quasiconvex and $f(a)$ is finite. Then*

- $h \in \tilde{K}(a) \Rightarrow -\infty \leq f'_-(a,h) \leq f'_+(a,h) \leq 0,$

- *$h \notin \tilde{K}(a) \Rightarrow 0 \leq f'_-(a,h) \leq f'_+(a,h) \leq +\infty$,*

- *If $\exists\, h$ such that $f'_-(a,h) < 0$ then $f'_-(a,d) < 0$ for all $d \in int(\tilde{K}(a))$,*

- *If $\exists\, h \in int(\tilde{K}(a))$ such that $f'_-(a,h) > -\infty$ then $f'_-(a,d) > -\infty$ for all d,*

- *If $\exists\, h$ such that $f'_+(a,h) < 0$ then $f'_+(a,d) < 0$ for all $d \in int(\tilde{K}(a))$,*

- *If $\exists\, h \in int(\tilde{K}(a))$ such that $f'_+(a,h) > -\infty$ then $f'_+(a,d) > -\infty$ for all d.*

Let C be a convex set and $f : C \to \mathbb{R}$ be differentiable at $a \in C$. The fonction f is said to be *pseudoconvex* at a on C if

$$y \in C \ \text{ and } f(y) < f(a) \Rightarrow \langle \nabla f(a), y - a \rangle < 0.$$

If f is pseudoconvex at any $a \in C$ then it is quasiconvex on C. Pseudoconvexity can be relaxed thanks to Dini-derivatives.

A function f defined on a convex set C finite at $a \in C$ is said to be D_+ *pseudoconvex* at a on C if

$$y \in C \ \text{ and } f(y) < f(a) \Rightarrow f'_+(a, y - a) < 0,$$

and D_- *pseudoconvex* at a on C if

$$y \in C \ \text{ and } f(y) < f(a) \Rightarrow f'_-(a, y - a) < 0.$$

Proposition 7.3 says that a quasiconvex function is D_+ *pseudoconvex* and D_- *pseudoconvex* at a as soon as $f'_+(a,h)$ and $f'_+(a,h)$ respectively are negative for some h. Then, according to the case

$$\tilde{K}(a) = \{ h : f'_+(a,h) < 0\},$$

or

$$\tilde{K}(a) = \{ h : f'_-(a,h) < 0\}.$$

If $\tilde{S}(a) = \{x : f(x) < f(a)\} \neq \emptyset$ and f is pseudoconvex at a in one of the senses above, then $a \in cl(\tilde{S}(a))$. That motivates another relaxation of pseudoconvexity, still weaker than the previous ones: a quasiconvex function is said to be S *pseudoconvex* if

$$\tilde{S}(a) \neq \emptyset \Rightarrow a \in cl(\tilde{S}(a))$$

The Dini-derivatives of a quasiconvex functions are not finite in general. Assume that f is quasiconvex, D_+ *pseudoconvex* at a and $|f'_+(a,h)| < \infty$ for all h. Since $f'_+(a,.)$ is quasiconvex and positively homogeneous of order 1, then the decomposition (8) can be applied. It follows that $f'_+(a,.)$ is the minimum of two finite convex functions, hence it is continuous except perhaps on the boundary of $\tilde{K}(a)$. Furthermore, it is convex on $\tilde{K}(a)$. The same conclusions do not hold for $f'_-(a,.)$ because not quasiconvex.

Generalized derivatives have been made popular with the Clarke approach. A good situation is when the function is locally Lipschitz. Unlike convex functions, quasiconvex functions are not locally Lipschitz on the interior of their domain: *Lipschitz properties do not belong to the essence of quasiconvexity.* Henceforth, our intimate opinion is that generalized derivatives for locally Lipschitz functions are quite inappropriate in quasiconvex analysis. However, because so many people have tried to adapt these generalized derivatives, we indicate that for quasiconvex functions, the Lipschitz condition holds if it holds in one direction. The result is as follows:

Theorem 7.1 *(Crouzeix [9]) Assume that f is quasiconvex on an open convex set C. Asume in addition that there exist a convex cone K, $h \in int(K)$ and a constant L such that*

- $K \subseteq \tilde{K}(x)$ *for all* $x \in C$,
- $|f(x+th) - f(x)| \leq L|t|$ *for all t and x such that both x and $x+th$ lie in C.*

Then there exists a constant $\hat{L}$ such that

$$|f(x) - f(y)| \leq \hat{L}\|x - y\| \text{ for all } x, y \in C.$$

Theorem 3.1 gives conditions for the existence of such a cone K.

8 The normal cone

Quasiconvex analysis finds its main applications in optimization. A general formulation of an optimisation problem is

$$\text{minimize } f(x) \text{ subject to } x \in C.$$

If C is convex and f is quasiconvex then we are faced with a quasiconvex optimization problem. A point $a \in C$ is an optimal solution if

$$\{x \;:\; f(x) < f(a)\} \cap C \;=\; \emptyset.$$

Both sets are convex. Hence we are concerned with vectors $a^\star \neq 0$ such that

$$\langle a^\star, x - a\rangle \leq 0 \leq \langle a^\star, z - a\rangle \;\; \forall x, z \text{ so that } z \in C, f(x) < f(a).$$

Such vectors belong to the closed convex cone

$$\tilde{N}(a) = \{\, a^\star \;:\; \langle a^\star, x - a\rangle \leq 0 \;\text{ for all } x \text{ so that } f(x) < f(a)\}$$

which is nothing else that the polar cone of $\tilde{K}(a)$, i.e.

$$\tilde{N}(a) = \{\, a^\star \;:\; \langle a^\star, d\rangle \leq 0 \;\text{ for all } d \in \tilde{K}(a)\}.$$

If f is convex and a belongs to the interior of the domain of f, then $\tilde{N}(a)$ is the projection on E of the set $T(a)$ in Equation (1). Hence, $\tilde{N}(a)$ is the closed convex cone generated by $\partial f(a)$, the Fenchel subgradient of f at a. $T(a)$ is the normal cone at point $(a, f(a))$ to the epigraph of f. When point a moves, then $(a, f(a))$ moves on the boundary of the epigraph which is convex. Hence the map T has continuity properties which afterwards implies continuity properties on the map ∂f. These properties are the tools used to analyse sensibility in convex programming.

The epigraph of a quasiconvex function is not convex and $T(a)$ cannot be used. However, the level sets are convex and therefore continuity should be considered not on the map T but on the map $\tilde{N}$.

Recall that a point-to-set map $M : E \to F$ is said to be *closed at a point* a if given a sequence $\{(a_n, b_n)\}_n$ converging to (a, b) with $b_n \in M(a_n)$ we have $b \in M(a)$.

The map is said to be *closed on* E if closed at any point $a \in E$. It is known that a map is closed on E if and only if its graph $G(M) = \{(x, y) : y \in M(x)\}$ is a closed subset of $E \times F$.

The map M is said to be *USC at a point* a if for all open set $\Omega \supseteq M(a)$ there is a neighbourhood V of a such that $\Omega \supseteq M(x)$ for all $x \in V$. We shall use the following well known characterization.

Proposition 8.1 *Let $M : E \to K$ be a point-to-set map where E is a metric space and K is a compact set. Assume that, for all x in a neighbourhood of a, the set $M(x)$ is compact and nonempty and the map M is closed at x. Then M is USC at a.*

Proposition 8.2 *Let $f : E \to [-\infty, +\infty]$ be quasiconvex. Assume that f is finite and lsc at a. Then $\tilde{N}$ is closed at a.*

Proof. Let $\{(a_n, b_n)\}_n$ be sequence converging to (a, b) with $b_n \in \tilde{N}(a_n)$. Let x be such that $f(x) < f(a)$. Then there is a neighbourhood V of a such that $f(x) < f(y)$ for all $y \in V$. For n large enough, $a_n \in V$ and therefore, because $b_n \in \tilde{N}(a_n)$,

$$\langle b_n, x - a_n \rangle \leq 0.$$

Hence, passing to the limit

$$\langle b, x - a \rangle \leq 0.$$

Because the inequality holds for all x with $f(x) < f(a)$, it results that $b \in \tilde{N}(a)$. ■

The normal cone $\tilde{N}$ is, of course, unbounded. Continuity with unbounded maps is sometimes difficult to handle. It is one of the reasons why the subgradient is used in sensibility in convex programming instead of the normal cone T at the epigraph.

We shall say that a point-to-set map $M : E \to F$ is *C-USC at a* (Borde-Crouzeix [1]) if there is a compact-valued map C such that C is USC at a and, for all x, $M(x)$ is the cone generated by $C(x)$, i.e.

$$M(x) = \{ y = \lambda d : \lambda \geq 0 \text{ and } d \in C(x) \}.$$

Theorem 8.1 *Let $f : E \to (-\infty, +\infty)$, f quasiconvex, $\lambda \in \mathbb{R}$ and $a \in E$ such that $int(S_\lambda(f)) \neq \emptyset$ and $a \notin cl(S_\lambda(f))$. Then $\tilde{N}$ is C-USC at a.*

Proof: By Theorem 3.1, there is K an open nonempty convex cone such that $K \subseteq \tilde{K}(x)$ for x close to a. Let $\bar{d} \in K$ be fixed. We define a map C by

$$C(x) \;=\; \{\, x^\star \in \tilde{N}(x) \;:\; \langle x^\star, \bar{d}\rangle = 1\}.$$

Then, for x close to a, $M(x)$ is generated by $C(x)$. Hence C is closed at x. Furthermore

$$C(x) \subseteq \bar{C} = \{\, x^\star \in K^o : \langle x^\star, \bar{d}\rangle = 1\}$$

where K^o is the polar cone of K. The set $\bar{C}$ is compact and the result follows. ■

If a convex function is differentiable, then its gradient is continuous. Now, assume that we are faced with a function f which is differentiable and quasiconvex. Let a be such $\nabla f(a) \neq 0$. Then

$$\tilde{N}(a) = \{\lambda \nabla f(a) \;:\; \lambda \geq 0\}.$$

It results from Theorem 8.1 that the map

$$x \to \frac{1}{\|\nabla f(x)\|} \nabla f(x)$$

is continuous at any x where the gradient does not vanishes. Recall that, for a convex function, the gradient itself is continuous, for a quasi convex function it is only the direction given by the gradient which is continuous.

Some people prefer to consider the normal cone

$$N(a) = \{\, a^\star \;:\; \langle a^\star, x - a\rangle \leq 0 \;\text{ for all } x \text{ so that } f(x) \leq f(a)\}$$

instead of $\tilde{N}(a)$. When f is pseudoconvex in one of the different senses we have given, the two cones coincide.

9 Are the generalized derivatives useful in quasiconvex analysis?

Assume that f is finite in a. A general formulation of a generalized derivative of f at a with respect to a direction d is as follows

$$\begin{aligned} f^!(a,d) \;&=\; \text{“some kind of limit”}\ \tfrac{1}{t}\,[f(x+th) - f(y)] \\ &\qquad \text{when } t \to 0_+,\ x,y \to a \text{ and } h \to d. \end{aligned} \tag{9}$$

The different ways by which the arguments x, y converge to a, h converges to d, the different orders to take the limits with respect to the arguments, the different types of limit (sup, inf, ...) give birth to many combinations quite appropriate to exercice the skills of a mathematician. For instance, the upper (lower) Dini-derivative corresponds to the case where $x = y = a$, $h = d$ and the limit is the limit sup (limit inf).

Once a generalized derivative is defined, a generalized subgradient is built as the closed convex set $\partial^! f(a)$ such that

$$\partial^! f(a) = \{ x^\star \,:\, \langle d, x^\star \rangle \leq f^!(a, d) \text{ for all } d \}.$$

Now, it is time to look at the uses of such a generalized derivative and such a subgradient. Because we deal with optimization problems, the first use concerns optimality conditions. We have seen that, in the quasiconvex setting, the normal cone $\tilde{N}(a)$ plays a fundamental role.Hence, the first property to be asked to generalized derivatives and/or generalized subgradients is that they allow to recover the cone $\tilde{N}(a)$. The second important use concerns sensibility. For that, a standard approach consists to consider some continuity on the subgradient, this continuity is provided in general by considering the convergences of the arguments x, y and h to a, a and d respectively in definition (9). *Is it really necessary to consider generalized derivatives and the associated subgradient and not to consider directly the normal cone* $\tilde{N}(a)$? We have seen, in the last section, that the normal cone has the wished continuity property.

Anyway, if we want to consider some generalized derivatives and their associate generalized subgradients, the simplest ones seem the best. Since, unlike in the convex case, directional derivatives cannot be considered, the best candidates are the Dini-derivatives. If f is pseudoconvex then Proposition 7.3 shows that $\tilde{N}(a)$ can be recovered from the Dini-derivatives. For simplicity we assume that these Dini-derivatives are finite. Let us define

$$\partial^u f(a) = \{x^\star : f'_+(a, d) \geq \langle x^\star, d \rangle \text{ for all } d \in \tilde{K}(a)\}.$$

and

$$\partial^l f(a) = \{x^\star : f'_-(a, d) \geq \langle x^\star, d \rangle \text{ for all } d \in \tilde{K}(a)\}.$$

Then, $\tilde{N}(a)$ is the cone generated by $\partial^u f(a)$ and/or $\partial^l f(a)$, hence it is thoroughly defined from the knowledge of these sets. Furthermore, because

$f'_+(a,.)$ is quasiconvex, for all $h \in int(\tilde{K}(a))$

$$f'_+(a,h) = \sup[\langle x^\star, h\rangle \,:\, x^\star \in \partial^u f(a)].$$

An exhaustive bibliography on generalized derivatives and generalized subgradients in quasiconvex programming is reference [14].

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