VECTOR OPTIMIZATION

(Lecture delivered at the summer school "Generalized Convexity and Monotonicity", August 25-28, 1999, Samos, Greece)

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CONTENTS

1. PRELIMINARIES

1. Partially ordered spaces

Let E be a space and $B \subseteq E \times E$ a relation on E. We say that B is a partial order on E if it is

- reflexive, i.e. $(x, x) \in B$ for every $x \in E$;
- transitive, i.e.$(x, y), (y, z) \in B$ imply $(x, z) \in B$.

In this note we shall deal with a special case where E is a topological vector space and equipped with a partial order B that is linear in the sense that $(x, y) \in B$ implies $(x + z, y + z), (tx, ty) \in B$ for all $z \in E$ and $t > 0$.

We shall write $x \geq y$ instead of $(x, y) \in B$.

Proposition. If $(\geq)$ is a linear partial order in E, then the set

$$C := \{x \in E : x \geq 0\}$$

is a convex cone in E.

Conversely, if C is a convex cone in E, then the relation

$$x \underset{C}{\geq} y \quad \text{if and only if } x - y \in C$$

is a linear partial order in E.

Proof Let $(\geq)$ be a linear partial order in E. Let $x \in C$. By the linearity, one has $tx \geq 0$ for all $t > 0$. Hence $tx \in C$ for $t > 0$. When $t = 0$, by the reflexivity one has $0 = 0x \in C$. This shows that C is a cone. This cone is convex because for $x, y \in C$ we have $x \geq 0$, $y \geq 0$, consequently $x + y \geq 0 + y \geq y \geq 0$ which means $x + y \in C$.

Conversely, assume that C is a convex cone in E. Since $0 \in C$, we have $x \underset{c}{\geq} x$ for all $x \in E$. This shows that $(\geq_c)$ is reflexive. Moreover, if $x - y \in C$ and $y - z \in C$, then by the convexity of C we obtain $x - z = x - y + y - z \in C$ or equivalently, $x \geq_c y$ and $y \geq_c z$ imply $x \geq_c z$. In this way $(\geq_c)$ is a partial order in E. It is linear because $x - y \in C$ implies $t(x - y) \in C$ for $t > 0$ and $(x + z) - (y + z) \in C$ for all $z \in E$, which means $x \geq_c y$ implies $tx \geq_c ty$ and $x + z \geq_c y + z$ for all $t > 0$, $z \in E$. The proof is complete. ■

Examples

1. Let $E = R^n$ and $C = R^n_+$ (the positive orthant). Then $(\geq_c)$ is the usual componentwise order, i.e. for $x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n)$ one has $x \geq_c y$ if and only if $x_i \geq y_i$, $i = 1, \cdots, n$.

This is a linear partial order.

2. The lexicographic order : Let C be the cone in R^n consisting of all vectors x, whose first nonzero component is positive. Then $(\geq_c)$ is a linear partial order. Actually this order is complete in the sense that any two elements of R^n are comparable (either $x \geq_c y$ or $y \geq_c x$).

3. The ubiquitous order : Let ℓ_0 denote the space of sequences whose terms are all zero except for a finite number. This is a normed space if we equip it with the max-norm. Let C be a cone consisting of sequences whose last nonzero component is positive. Then the order generated by C is a linear partial order in ℓ_0. The cone C is called ubiquitous because of the following property : for each $x \in \ell_0$, there exists $y \in C$ such that $[y, x) \subseteq C$.

2. Correct cones

Let C be a convex cone in a topological vector space E. We shall make use of the following notations : $\ell(C) := C \cap -C$ (the linear part of C) ; int C (the interior of C), cl C (the closure of C).

For a subset $A \subseteq E$, A^c denotes the complement of A in E, i.e. $A^c = E \setminus A$.

Definition. We say that the cone C is

i) pointed if $\ell(C) = \{0\}$

ii) correct if cl $C + C \setminus \ell(C) \setminus \subseteq C$
or equivalently cl $C + C \setminus \ell(C) \subseteq C \setminus \ell(C)$.

Note that the cone R^n_+ is pointed and correct, while the lexicographic cone and the ubiquitous cone are pointed and not correct.

Proposition. Each of the following conditions is sufficient for C to be correct :

i) C is closed

ii) $C \setminus \ell(C)$ is open

iii) C consists of the origin and an intersection of half-spaces that are either open or closed.

Proof. If C is closed, then cl $C = C$ and the correctness of C follows from the convexity.

If $C \setminus \ell(C)$ is open, then int $C \neq \emptyset$ and $C \setminus \ell(C) =$ int C. Consequently, cl $C + C \setminus \ell(C) =$ cl C + int $C \subseteq C$ which shows that C is correct.

Finally, let $C = \{0\} \cup \{\bigcap_{\lambda \in \Lambda} H_\lambda\}$ where each H_λ is a half-space that is either closed or open. If all of H_λ are closed, then C is closed. By the first part, C is correct. If one of H_λ is open, then $\ell(C) = \{0\}$ and $b \in C \setminus \ell(C)$ if and only if

$b \in H_\lambda$ for all $\lambda \in \wedge$. Moreover, an element $a \in$ cl C if and only if $a \in$ cl H_λ for all $\lambda \in \wedge$. This and the fact that cl $H_\lambda + H_\lambda \subseteq H_\lambda$ independant of whether H_λ is open or closed, imply that $a + b \in C$ whenever $a \in$ cl C and $b \in C \setminus \ell(C)$. Hence C is correct. ■

3. C-complete Sets

Let E be a topological vector space and C a convex cone in E. We shall write $x > y$ by understanding $x - y \in C \setminus \ell(C)$.

Let $\{x_i\}_{i \in I}$ be a net in E. It is said to be decreasing if $x_i > x_j$ for $i < j$.

Definition. A set $A \subseteq E$ is said to be C-complete (resp. strongly C-complete) if it has no covering of the form

$$\{(x_i - \text{ cl } C)^c : i \in I\} \quad \big(\text{resp. } \{(x_i - C)^c : i \in I\}\big)$$

where $\{x_i\}_{i \in I}$ is a decreasing net in A.

We note that every strongly C-complete set is C-complete. The converse is not always true except for the case where C is a closed cone.

Below we give some sufficient conditions for a set to be C-complete.

Proposition 1. Every compact set is C complete. In particular every weakly compact set in a locally convex space is C-complete.

Proof. Let A be a compact set in E and let $\{x_i\}_{i \in I}$ be a decreasing net in A. If the family $\{(x_i - \text{ cl } C)^c : i \in I\}$ covers A, then it is an open covering of A. Since A is compact, one may extract a finite subcovering, say $\{(x_{i_\ell} - \text{ cl } C)^c : \ell = 1, \cdots, k\}$. Let $i_0 \in I$ such that $i_0 \geq i_\ell$ for $\ell = 1, \cdots, k$. Then one has

$$x_{i_0} < x_{i_\ell}\ , \ \ell = 1, \cdots, k\ .$$

On the other hand, there exists $j \in \{1, \cdots, k\}$ such that $x_{i_0} \in (x_{i_j} - \text{ cl } C)^c$. This implies $x_{i_0} \not< x_{i_j}$, a contradiction. Thus A is C-complete.

Now if E is locally convex, then cl C is also closed in the weak topology. It remains to apply the above reasonning for the weak topology. ■

Proposition 2. If E is a finite dimensional space, then every compact set is strongly C-complete.

Proof. We prove this proposition by induction on the dimension of C. If dim $C = 1$, then either C is a straight line or a half-line. In both cases C is closed. By Proposition 1, every compact set is C-complete. By a remark made before Proposition 1, the set is strongly C-complet as well. Assuming the conclusion legal for dim $C < m$, we show it for dim $C = m$. Suppose to the contrary that a compact set $A \subseteq R^n$ is not strongly C-complete, that is there exists a decreasing net $\{x_i\}_{i\in I} \subseteq A$ such that $\{x_i - C)^c : i \in I\}$ is a covering of A. Since the space is of finite dimension, we may assume that the net is a sequence $\{x_i\}_{i\geq 1}$ that converges to some point $x_* \in A$. There exists i_0 such that $x_* \in (x_{i_0} - C)^c$, or equivalently $x_* \notin x_{i_0} - C$. It follows that $x_* \notin x_i - C$ for all $i \geq i_0$. By this we may assume $i_0 = 1$. Denote by L the smallest linear subspace containing $x_i - x_1$, $i = 2, 3, \cdots$. We want to show that $L \cap$ ri $C = \emptyset$. In fact, let $x \in L$. Then x can be expressed as a linear combination

$$x = \sum_{i=1}^{\ell} t_j (x_{i_j} - x_1)$$

with $t_j \neq 0$, $\quad i_j \in \{1, 2, \cdots\}$ and $\quad i_1 < i_2 < \cdots$. We prove $x \notin$ ri C by induction on ℓ.

If $\ell = 1$, then $x = t_1(x_{i_1} - x_1)$. Since $x_{i_1} - x_1 \in -C \setminus \ell(C)$, one has $x_{i_1} - x_1 \notin$ ri C , consequently $x \in$ ri C is possible only when $t_1 < 0$. Then $x_{i_1} \in x_1-$ ri C by supposing $x \in$ ri C. Moreover as $x_i \in x_{i_1} - C$ for $i \geq i_1$, one obtains $x_* \in x_{i_1}-$ cl C. Consequently,

$$x_* \in x_{i_1} - \text{ cl } C \subseteq x_1 - \text{ cl } C - \text{ ri } C \subseteq x_1 - C \ ,$$

a contradiction.

Assuming that $x \notin$ ri C whenever x is a linear combination of $\ell \geq 1$ terms, we prove $x \notin$ ri C when

$$x = \sum_{j=1}^{\ell+1} t_j (x_{i_j} - x_1) \ .$$

Suppose to the contrary that $x \in$ ri C. If $t_\ell > 0$, we have

$$x - t_\ell (x_{i_\ell} - x_1) = \sum_{\substack{j=1 \\ j\neq \ell}}^{\ell+1} t_j (x_{i_j} - x_1) \ .$$

The vector in the left hand side belongs to ri C because $-t_\ell(x_{i_\ell} - x_1) \in C \setminus \ell(C)$ and $x \in$ ri C, while the vector in the right hand side is a combination of ℓ terms and is not in ri C by induction. This contradiction shows that $t_\ell < 0$. In this cas we obtain

$$x - t_\ell (x_{i_\ell} - x_{i_{l+1}}) = \sum_{j=1}^{\ell-1} t_j (x_{i_j} - x_1) + (t_\ell + t_{\ell+1}) (x_{i_{\ell+1}} - x_1) \ .$$

Since $x_{i_\ell} - x_{i_{\ell+1}} \in C$ and $t_\ell < 0$, the vector in the left hand side belongs to ri C, while the vector in the right hand side does not belong to ri C. The contradiction shows that $L \cap$ ri $C = \emptyset$.

Now we separate L and ri C by a hyperplane $H : H \supseteq L$ and $H\cap$ ri $C = \emptyset$. Putting $C_1 := C \cap H$ we see that C_1 is a convex cone with dim $C_1 <$ dim C. Moreover, as $C_1 \subseteq C$, one has $\left(x_i - C_1\right)^c \supseteq \left(x_i - C\right)^c$ and consequently the family $\left\{(x_i - C_1)^c : i \in I\right\}$ covers A. By induction on the dimension of the cone, the set A has no coverings of the above form. By this A is strongly C-complete.

■

II.EFFICIENT POINTS AND EXISTENCE CRITERIA

4. Efficient Points

Definition. Let A be a subset of a topological vector space E equiped with a linear partial order that is generated by a convex cone C. We say that a point $a \in A$ is

i) an ideal point of A if $x \geq a$ for every $x \in A$.

The set of all ideal points of A is denoted by IMin A or IMin(A/C).

ii) an efficient point of A if whenever $a \geq x$ for some $x \in A$ one has $x \geq a$.

The set of all efficient points of A is denoted by Min A or Min(A/C).
Sometimes one is interested also in the set of efficient points with respect to the ordering generated by the cone $\{0\}\cup$ int C if int $C \neq \emptyset$. This is the set of weakly efficient points and denoted by WMin A or WMin(A/C). If there exists a convex cone $K \neq E$ with int $K \supseteq C \setminus \ell(C)$, such that $a \in$ Min(A/K), then we call it properly efficient. The set of all properly efficient points of A is denoted by PrMin A or PrMin(A/C).
Note that there are some other definitions of proper efficient points. They coincide with the one we gave in the case where A is a convex set in a finite dimensional space.

Exemple

1. Let $A = \{(x, y) \in R^2 : x^2 + y^2 \leq 1 \quad \text{or} \quad x \geq 0,\ |y| \leq 1\} \subseteq R^2$ and let $C = R^2_+$. Then

$$
\begin{aligned}
\text{IMin } A &= \emptyset \\
\text{PrMin } A &= \{(x, y) \in R^2 : \quad x^2 + y^2 = 1\ ,\ x < 0,\ y < 0\} \\
\text{Min } A &= \{(x, y) \in R^2 : \quad x^2 + y^2 = 1,\ x \leq 0,\ y \leq 0\} \\
\text{WMin } A &= \text{Min } A \cup \{(x, -1) : x \geq 0\}\ .
\end{aligned}
$$

2. For $E = \ell_0$ (Example 3 of 1), C the ubiquitous cone, the unit ball has no efficient points.

Below is an equivalent definition of efficient points.

Proposition 1. Let $A \subseteq E$. Then

i) $a \in I$Min A if and only if $a \in A$ and $A \subseteq a + C$;

ii) $a \in \text{Min } A$ if and only if $a \in A$ and $A \cap (a - C) \subseteq a + \ell(C)$.

In other words $a \in \text{Min } A$, if and only if $a \in A$ and there is no $y \in A$ with $a > y$;

iii) $a \in W\text{Min } A$ if and only if $a \in A$ and $A \cap (a - \text{ int } C) = \emptyset$.

Proof. These conclusions are direct consequences of the definition ■

The relationship between the different concepts of efficiency is seen in the next result. We suppose always that $C \neq E$.

Proposition 2. For every nonempty set $A \subseteq E$ one has

$$\text{PrMin } A \subseteq \text{ Min } A \subseteq W\text{Min } A \, .$$

Moreover, if $I\text{Min } A \neq \emptyset$, then $I\text{Min } A = \text{Min } A$ and this set is a singleton whenever C is pointed.

Proof. For the first inclusion let $x \in \text{PrMin } A$. If $x \notin \text{Min } A$, then there is $y \in A$ with $x - y \in C \setminus \ell(C)$. Let $K \notin E$ be a convex cone with int $K \supseteq C \setminus \ell(C)$ and $x \in \text{Min}(A/K)$. Then $x - y \in \text{int } K \subseteq K \setminus \ell(K)$ which contradicts $x \in \text{Min}$ (A/K). Next, let $x \in \text{Min } A$. If $x \notin W \text{ Min } A$, then by Proposition 1, ii), there exists $y \in A$ such that $x - y \in \text{int } C$. As $C \neq E$, int $C \subseteq C \setminus \ell(C)$ and we have $x - y \in C \setminus \ell(C)$, a contradiction with the fact that $x \in \text{ Min} A$.
Finally, let $x \in I\text{Min } A$. It follows that $x \in \text{Min } A$. Let $y \in \text{Min } A$. Then $y \geq x$, hence $x \geq y$. For any $z \in A$ one has $z \geq x$ because $x \in I\text{Min } A$. Consequently $z \geq y$, which shows that $y \in I\text{Min } A$. By this $I\text{Min } A = \text{ Min} A$. If in addition C is pointed, then $x \geq y$ and $y \geq x$ imply $x = y$. Thus $I\text{Min } A$ is a singleton. ■

If the space E is equipped with two orders then the relationship between efficiencies with respect to these cones is expressed by the next proposition.

Proposition 3. Assume that K is a pointed convex cone with $C \subseteq K$. Then we have

i) $I \text{ Min}(A/K) = I \text{ Min}(A/C)$ provided $I \text{ Min}(A/C)$ is nonempty ;

ii) $\text{Min}(A/K) \subseteq \text{Min } (A/C)$ (similarly for W Min and Pr Min).

Proof. Observe that C is pointed. By Proposition 2, if $I \text{ Min}(A/C)$ is nonempty, it is a singleton, say $\{x\}$. In view of Proposition 1, $A \subseteq x + C$. It follows that $A \subseteq x + K$ which means $I \text{ Min}(A/K) = \{x\}$.

Now let $x \in \text{Min } (A/K)$. By Proposition 1, $A \cap (x - K) = \{x\}$. Therefore $A \cap (x - C) = \{x\}$, which implies $x \in \text{Min}(A/C)$. The proof for W Min and PrMin is analogous. ■

Note that the above result is no longer true if K is not pointed except for the particular case where K is a closed half-space.

We shall denote by $A_x := A \cap (x - C)$ for $x \in E$ and call it a section of A at x.

<u>Proposition 4</u>. Let $x \in E$ with $A_x \neq \emptyset$. The following assertions hold

i) IMin $A_x \subseteq I$ Min A if IMin $A \neq \emptyset$;

ii) Min $A_x \subseteq$ Min A (similarly for W Min).

<u>Proof</u>. For the first inclusion, let $y \in I$Min A_x and $z \in I$Min A. We have $A_x \subseteq y + C$ and $A \subseteq z + C$. Then $z \in A_x$ and $z - y \in \ell(C)$. This implies

$$A \subseteq z + C = z - y + y + C = y + \ell(C) + C = y + C$$

which shows $y \in I$Min A.

Next, assume $y \in$ Min A_x. By Proposition 1, we have $A_x \cap (y - C) \subseteq y + \ell(C)$. Since $y - C \subseteq x - C$, we obtain

$$A \cap (y - C) \subseteq A \cap (y - C) \cap (x - C) \subseteq A_x \cap (y - C) \subseteq y + \ell(C)$$

which shows that $y \in$ Min A.

The proof for WMin is analogous. ■

Remark that the inclusion Pr Min $A_x \subseteq$ Pr Min A is not true in general except for very specific cases.

5. <u>Existence criteria</u>

<u>Theorem 1</u>. Le A be a nonempty set in E. Then Min $A \neq \emptyset$ if and only if there is $x \in E$ such that A_x is nonempty and strongly C-complete.

<u>Proof</u>. The necessity is obvious because by taking $x \in$ Min A, the selection A_x is nonempty and has no decreasing nets, hence strongly C-complete.

For the sufficiency, suppose to the contrary that for some $x \in E$, the selection A_x is nonempty and strongly C-complete, but Min $A = \emptyset$. Denote by $\mathcal{P}$ the set of all decreasing nets in A_x and introduce a partial order on $\mathcal{P}$ by inclusion, i.e. for $a, b \in \mathcal{P}$ one writes $a \geq b$ if and only if $b \subseteq a$ as sets. We observe that $\mathcal{P}$ is nonempty because Min $A = \emptyset$ and the above introduced order is a partial order on $\mathcal{P}$. Now we prove that $\mathcal{P}$ satisfies the hypothesis of Zorn's lemma : every chain $X = \{a_\lambda : \lambda \in \Lambda\} \subseteq \mathcal{P}$ has an upper bound. Indeed, denote by $\mathcal{B}$ the family of all finite subsets of Λ. For each $B \in \mathcal{B}$ we set

$$a_B := \bigcup_{\lambda \in B} a_\lambda \ .$$

It is evident that $a_B \in \mathcal{P}$. Now we put

$$a_0 = \cup\{a_B : B \in \mathcal{B}\} \ .$$

Let I_0 be the index set consisting of all elemnts of a a_0 with $\alpha > \beta$ if $\beta \underset{C}{>} \alpha$ being considered as elements of a_0. In other words the index set order is defined by the cone $(-C \setminus \ell(C)) \cup \{0\}$. Then I_0 is a directed index set because for $\alpha, \beta \in I_0$ there exist $B_1, B_2 \in \mathcal{B}$ such that $\alpha \in a_{B_1}$ and $\beta \in a_{B_2}$. Taking $B = B_1 \cup B_2$ we see that $\alpha, \beta \in a_B$. Since a_B is a decreasing net, there is $\gamma \in a_B$ such that $\alpha \underset{C}{>} \gamma$ and $\beta \underset{C}{>} \gamma$. Then $\gamma \in I_0$ with $\gamma > \alpha$ and $\gamma > \beta$. Moreover, it is evident that $a_0 \geq a$ for all $a \in X$. Hence a_0 is an upper bound of X. Now we apply Zorn's lemma to obtain a maximal element a_*, say $a_* = \{x_i\}_{i \in I} \in \mathcal{P}$.we claim that the family $\{(x_i - C)^c : i \in I\}$ is a covering of A. Indeed, if not, there is $y \in A$ such that $y \notin (x_i - C)^c$ for all $i \in I$, or equivalently $y \in x_i - C$ for all $i \in I$. Since Min $A = \emptyset$, for this y there exists $z \in y - C \setminus \ell(C)$. It follows that

$$z \in x_i - C - C \setminus \ell(C) \subseteq x_i - C \setminus \ell(C) \ .$$

In other words $z \in A_x$ and $z \underset{C}{<} x_i$ for all $i \in I$. This contradicts the maximality of a_*. In this way the family $\{(x_i - C)^c : i \in I\}$ covers A_x. This is impossible because A_x is strongly C-complete. The proof is complete. ■

<u>Theorem 2</u>. Assume that A is a nonempty set in E and C is correct. Then Min $A = \emptyset$ if and only if there is $x \in E$ such that A_x is nonempty and C-complete.

<u>Proof</u>. Proceed in the same way as in the proof of the preceding theorem by using the following characterization of a correct cone

$$\mathrm{cl}\ C + C \setminus \ell(C) \subseteq C \setminus \ell(C)$$

in order to obtain $z \underset{C}{>} x_i$ for all $i \in I$. ■

<u>Corollary.</u> If A is a nonempty compact set in a finite dimensional space, then Min $A \neq \emptyset$ whatever the cone C be.

If A is a nonempty compact set in an infinite dimensional space and the cone C is closed, then Min $A \neq \emptyset$.

<u>Proof</u>. Invoke Theorems 1,2 above and Proposition 1,2 of Section 2. ■

Note that in an infinite dimensional space a compact set may have no efficient points if the cone C is not correct. To see this, consider the following example. Let E be ℓ_0 and C be the ubiquitous cone (Example 3 of Section 1).

Let $x_0 = (1, 0, 0, \cdots)$, $x_n = (1, -\frac{1}{2^n}, \cdots, -\frac{1}{2^n}, 0, \cdots 0)$ and $A = \{x_i : i = 0, 1, 2, \cdots\}$. It is evident that $\lim_{n \to \infty} x_n = x_0$. Hence A is a compact set. Despite of this, Min $A = \emptyset$ because $x_0 > x_1 > x_2 \cdots$.

III. OPTIMALITY CONDITIONS

6. Differentiable Problems

Let us consider the following vector problem (VP) :

$$\text{Min } f(x)$$
$$g(x) \leq 0$$
$$h(x) = 0$$

where f, g and h are functions from X to Y, Z and W respectively with X, Y, Z and W Banach spaces. We assume that Y and Z are partially ordered by convex pointed cones C_y and C_z having nonempty interiors. The above problem means finding a point $x_0 \in X$ (called an efficient solution) such that $f(x_0)$ is an efficient point of the set $\{f(x) \in Y : x \in X, g(x) \leq 0 \ , \ h(x) = 0\}$. A weakly efficient solution is defined in a similar way. A solution is local if one restricts the problem on a neighborhood of this point. In this section we shall derive a necessary condition for local weakly efficient solutions. Two classic results of analysis will be needed :

1. Mean Value Theorem (MVT) : If f is Gateaux differentiable on X, then for each $a, b \in X$ one has

$$||f(b) - f(a)|| \leq \sup\{||f'(c)|| \cdot ||b - a|| : c \in [a, b]\} \ .$$

2. Open Mapping Theorem (Lyusternik's Theorem) : If h is Fréchet differentiable with h' continuous at x_0 and if $h'(x_0)$ is surjective, then the tangent cone to the set $M := \{x \in X : h(x) = 0\}$ at $x_0 \in M$ defined by

$$T_M(x_0 := \{v \in K : v = \lim_{i \to \infty} t_i(x_i - x_0), \ t_i > 0, \quad x_i \longrightarrow x_0, \ x_i \in M\}$$

coincides with Ker $h'(x_0)$.

We recall also that the positive polar cone of a cone $C \subseteq Y$ is defined by

$$C' := \{\xi \in Y' : \prec \xi, y \succ \geq 0 \quad \text{for all } y \in C\}$$

where Y' denotes the topological dual space of Y.

Theorem. Assume that f, g and h are Frechet differentiable with f' and g' bounded and h' continuous in a neighborhood of x_0. If x_0 is a local weakly efficient solution of (VP), then there exist multipliers $(\xi, \theta, \gamma) \in (C_Y, C_Z, \{0\})' \setminus \{0\}$ such that

$$\xi f'(x_0) + \theta g'(x_0) + \gamma h'(x_0) = 0 \ , \qquad \theta g(x_0) = 0 \ .$$

Proof. Assume first that $h'(x_0)$ is not surjective, i.e. $h'(x_0)(X)$ is a proper subspace of W. Then there exists a nonzero functionnal $\gamma \in W' \setminus \{0\}$ such that

$$\prec \gamma, h'(x_0)(u) \succ = 0 \quad \text{for all } u \in X \ .$$

This implies $\gamma h'(x_0) = 0$. Now setting $\xi = 0$ and $\theta = 0$ we obtain multipliers (ξ, θ, γ) as requested.

Now consider the case where $h'(x_0)$ is surjective. We want to show that

$$(f'(x_0), g'(x_0), h'(x_0)(X) \cap (-\text{int } C_Y, -g(x_0) - \text{int } C_Z, \{0\}) = \emptyset \ . \qquad (1)$$

In fact, if this intersection is not empty, then there is a vector $u \in X$ with $||u|| = 1$ such that

$$\begin{aligned} & f'(x_0)(u) \in -\text{ int } C_Y \\ & g'(x_0)(u) \in -g(x_0) - \text{ int } C_Z \\ & h'(x_0)(u) = 0. \end{aligned}$$

Applying Lyusternik's theorem we find $x_i \in M \setminus \{x_0\}$ such that $\{x_i\}$ converges to x_0 and $\{u_i\}$ with $u_i = (x_i - x_0)/||x_i - x_0||$, converges to u. Note that as f' is bounded in a neighborhood of x_0, in view of (MVT) we have the following estimate :

$$\lim \frac{f(x_i) - f(x_0)}{||x_i - x_0||} = f'(x_0)(u) \ .$$

Hence, for i sufficiently large we obtain

$$f(x_i) - f(x_0) \in -\text{ int } C_Y \ .$$

Similarly, for i sufficiently large we have

$$\frac{g(x_i) - g(x_0)}{||x_i - x_0||} \in -g(x_0) - \text{ int } K \ .$$

Since $||x_i - x_0||$ tends to 0 as i tends to ∞ the above implies

$$g(x_i) \in (1 - ||x_i - x_0||) g(x_0) - \text{ int } C_Z \subseteq -C_Z$$

for i sufficiently large. This and the fact that $h(x_i) = 0$ (because $x_i \in M$), together with (2) show that x_0 is not a local weakly efficient solution of (VP), a contradiction.

In this way (1) is true. We separate those convex sets of (1) by a linear functional $(\xi, \theta, \gamma) \in (Y, Z, W)' \setminus \{0\}$:

$$\xi f'(x_0)(u) + \theta[g'(x_0)(v) + g(x_0)] + \gamma h'(x_0)(w) \geq \prec \xi, -c \succ + \prec \theta, -k \succ$$

for all $u \in Y,\ v \in Z,\ w \in W$,
$c \in C_Y,\ k \in C_Z$.
It follows from the above inequality that
$\xi \in C',\ \theta \in K',\ \gamma \in W'$ and $\theta g(x_0)0$. Remember that $g(x_0) \in -K$, hence $\theta g(x_0) = 0$. Moreover, one has

$$\xi f'(0)(u) + \theta g'(x_0)(v) + \gamma h'(x_0)(w) \geq 0$$

for all $u \in Y$, $v \in Z$, $w \in W$ which implies

$$\xi f'(x_0) + \theta g'(x_0) + \gamma h'(x_0) = 0$$

as required. ∎

7. Lipschitz continuous problems

In this section we consider the problem (VP) in finite dimensional spaces that is we suppose that $X = R^n$, $Y = R^m$, $Z = R^k$ and $W = R^\ell$.
Recall that Clarke's generalized Jacobian of a locally Lipschitz function f from R^n to R^m is defined by

$$\partial f(x) := \overline{\text{co}}\{\lim_{i \to \infty} f'(x_i) : x_i \to x \ ,\ f'(x_i) \text{ exists}\}$$

where $\overline{\text{co}}$ denotes the closed convex hull.
We shall use the following properties of generalized Jacobian :

i) $\partial f(x)$ is compact, convex ;

ii) The set valued map $x \longmapsto \partial f(x)$ is upper semi-continuous ;

iii) In the case $m = 1$

$$\partial(f_1 + f_2)(x) \subseteq \partial f_1(x) + \partial f_2(x) \ ,$$

$\partial(\max_{\alpha \in T} f_\alpha(x) = \partial f_{\alpha_0}(x)$ if α_0 is the unique index where the maximum is attained.

$0 \in \partial f(x)$ if x is a local minimum of f.

iv) The mean value theorem : for $a, b, \in R^n$, one has

$$f(b) - f(a) \in \overline{\text{co}}\{M(b-a) : M \in \partial f(c)\ ,\ c \in [a,b]\}\ .$$

We shall also use Ekeland's variational principle :

Let φ be a lower semicontinuous function on R^n. If $\varphi(x_0) \leq \inf \varphi + \zeta$ for some $\zeta > 0$, then there is $x_\zeta \in R^n$ such that

$$\begin{array}{lll} ||x_\zeta - x_0|| & \leq \sqrt{\zeta} & \\ \varphi(x_\zeta) & \leq \varphi(x_0) & \\ \varphi(x_\zeta) & < \varphi(x) + \sqrt{\zeta}||x - x_\zeta|| & \text{for all } x \neq x_\zeta \ . \end{array}$$

Theorem. Assume that f, g and h are Lipschitz continuous and x_0 is a weakly efficient solution of (VP). Then there exist multipliers $(\xi, \theta, \gamma) \in (C_Y, C_Z, \{0\})' \setminus \{0\}$ such that

$$0 \in \partial(\xi f + \theta g + \gamma h)(x_0)$$

$$\theta g(x_0) = 0 \ .$$

Proof. Let $\lambda = (\xi, \theta, \gamma) \in (C_Y, C_Z, \{0\})' \setminus \{0\}$ and $T = \{\lambda : ||\lambda|| = 1\}$. Let $e \in \text{int } C_Y$ such that

$$1 = \max \{\prec \xi, e \succ : \xi \in C'_Y, ||\xi|| = 1\} \ .$$

For $\zeta > 0$ set

$$H_\zeta(x) := (f(x) - f(x_0) + \zeta e, g(x), h(x)\}$$

and consider the function

$$F_\zeta(x) := \max_{\lambda \in T} \prec \lambda, H_\zeta(x) \succ \ . \tag{1}$$

It is evident that $F_\zeta(x)$ is Lipschitz continuous. We want to apply Ekeland's principle to obtain a point x_ζ that minimizes the function $F_\zeta(x) + \sqrt{\zeta}||x - x_\zeta||$. To this purpose, we prove that $F_\zeta(x) > 0$ for all $x \in R^n$. Indeed, if not, i.e. $F_\zeta(x) \leq 0$ for some x, then $g(x) \leq 0$, $h(x) = 0$ and

$$\prec \xi, f(x) - f(x_0) \succ < 0 \quad \text{for all } \xi \in C'_Y \setminus \{0\} \ .$$

This means that x is a feasible solution and satisfies

$$f(x) - f(x_0) \in \text{ int } C \ ,$$

a contradiction to the optimality of x_0. In this way $F_\zeta(x) > 0$. We obtain then

$$F_\zeta(x_0) = \zeta \leq \inf_x F_\zeta(x) + \zeta \ .$$

According to Ekeland's principle, there is x_ζ such that

$$\begin{array}{l} ||x_\zeta - x_0|| \leq \sqrt{\zeta} \\ F_\zeta(x_\zeta) < F_\zeta(x) + \sqrt{\zeta}||x - x_\zeta||, \quad \text{for } x \neq x_\zeta \ . \end{array}$$

In other words, x_ζ is a minimum of the function $F_\zeta(x) + \sqrt{\zeta}||x - x_\zeta||$. Consequently we have

$$0 \in \partial(F_\zeta(x) + \sqrt{\zeta}||x - x_\zeta||)(x_\zeta) \subseteq \partial F_\zeta(x_\zeta) + \sqrt{\zeta}\, B(0,1) \tag{2}$$

where $B(0,1)$ denotes the unit ball in R^n (it is Clark's subdifferential of the function $x \longmapsto ||x - x_\zeta||$ at x_ζ). To calculate the subdifferential $\partial F_\zeta(x_\zeta)$ we make the following observation : Since $F_\zeta(x) > 0$, the vector $H_\zeta(x_\zeta) \neq 0$, hence the linear function $\lambda \longmapsto \prec \lambda, H_\zeta(x_\zeta) \succ$ attains its maximum at a unique point $\lambda_\zeta \in T$ on T (This is so because if that function has two distinct minima λ_1 and λ_2 on T, then at $\lambda = (\lambda_1 + \lambda_2)/||\lambda_1 + \lambda_2|| \in T$ one has

$$\prec \lambda, H_\zeta(x_\zeta) \succ = \frac{2}{||\lambda_1 + \lambda_2||} \prec \lambda_1, H_\zeta(x_\zeta) \succ > \prec \lambda_1, H_\zeta(x_\zeta) \succ$$

because $||\lambda_1 + \lambda_2|| < ||\lambda_1|| + ||\lambda_2|| \leq 2$, a contradiction, [Note that $\lambda_1 + \lambda_2 \neq 0$].)
We obtain

$$\partial F_\zeta(x_\zeta) = \partial \prec \lambda_\zeta, H_\zeta(x_\zeta) \succ = \partial(\xi_\zeta f + \theta_\zeta g + \gamma_\zeta h)(x_\zeta)\ .$$

Observe that in Ekeland's principle, if $\zeta \longrightarrow 0$, then $x_\zeta \longrightarrow x_0$. Moreover, as $H_\zeta(x_\zeta) \longrightarrow (0, g(x_0), h(x_0))$, one has $\lambda_\zeta \longrightarrow \lambda_0 \in T$ for some λ_0. Further, since $\partial \prec \lambda_\zeta, H_\zeta(x_\zeta) \succ = \partial \prec \lambda_\zeta, H_0(x_\zeta) \succ$, the upper semicontinuity of the subdifferential map

$$\prec \lambda, x \succ \longmapsto \partial \prec \lambda, H_0(x) \succ$$

and (2) show that

$$0 \in \partial \prec \lambda_0, H_0(x_0) \succ = \partial(\xi f + \theta g + \gamma h)(x_0)\ .$$

Finally, to see $\theta g(x_0) = 0$, it suffices to note that as $F_\zeta(x) > 0$, by letting $\zeta \longrightarrow 0$, we obtain $\theta g(x_0) \geq 0$. On the other hand $g(x_0) \in -C_Z$ and $\theta \in C'_Z$ imply $\theta g(x_0) \leq 0$. Thus, $\theta g(x_0) = 0$ and the proof is complete. ∎

Remark that the condition presented in the above theorem is useful if the first multiplier $\xi \neq 0$. One can guarantee this by imposing certain constraint qualification for instance all the matrices $N \in \partial h(x_0)$ has rank equal to ℓ and there exists $u \in \cap\{\ker N : N \in \partial h(x_0)\}$ such that $M(u) \in -\operatorname{int} C_Z$ for all $M \in \partial g(x_0)$.

8. Convex Problems

Consider the following convex problem (VP)

$$\begin{gathered}\operatorname{Min} f(x)\\ g(x) \leq 0\end{gathered}$$

where f is a convex function from R^n to R^m, g is a convex function from R^n to R^k. We recall that f is convex if for $\lambda \in (0,1)$, $x, y \in R^n$ one has

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \ .$$

The ordering cone $C_Y \subseteq R^m$ is supposed to be convex, closed pointed with nonempty interior and the ordering cone $C_z \subseteq R^k$ is supposed to be convex, closed. One can show that for a convex problem every local efficient solution is a global efficient solution. For a convex problem we have the following sufficient condition.

Theorem. Assume that f and g are convex and there exist multiplicators $(\xi, \theta) \in (C_Y, C_Z)' \setminus [0\}$ such that

$$0 \in \partial(\xi f)(x_0) + \partial(\theta g)(x_0)$$

$$\theta g(x_0) = 0 \ .$$

Then x_0 is an efficient (resp. weakly efficient solution of (VP) if $\xi \in \text{int } C_Y$ (resp. $\xi \in C' \setminus \{0\}$).

Proof. We prove the case of weakly efficient solutions. The other case is similar. Suppose to the contrary that x_0 is not weakly efficient, i.e. there exists a feasible solution $x \in R^n$ $(g(x) \leq 0)$ such that

$$f(x) - f(x_0) \in - \text{int } C \ .$$

On the one hand we have

$$\max_{\lambda \in \partial(\xi f)(x_0)} \prec \lambda, x - x_0 \succ = (\xi f)'(x_0, x - x_0) \leq \xi f(x) - \xi f(x_0) < 0$$

because $\xi \in C_Y' \setminus \{0\}$, where $(\xi f)'(x_0, x - x_0)$ denotes the directional derivative of the convex scalar function ξf at x_0 in direction $x - x_0$. [Note that $\partial(\xi f)(x_0)$ coincides with the convex analysis subdifferential of ξf at x_0]. On the other hand for $g(x)$ one has

$$\max_{\lambda \in \partial(\theta g)(x_0)} \prec \lambda, x - x_0 \succ = (\theta g)'(x_0, x - x_0) \leq \theta g(x) - \theta g(x_0) \leq 0$$

because $\theta g(x_0) = 0$ and $g(x) \in -C_Z$, $\theta \in C_Z'$. It follows from the above inequalities that

$$\max_{\lambda \in \partial(\xi f)(x_0) + \partial(\theta g)(x_0)} \prec \lambda, x - x_0 \succ < 0$$

which shows $0 \notin \partial(\xi f)(x_0) + \partial(\theta g)(x_0)$, a contradiction. ∎

9. Second order conditions

For the sake of simplicity let us present second order conditions for an unconstrained problem (VP)

$$\operatorname*{Min}_{x \in R^n} \; f(x)$$

where f is a function from R^n to R^m and R^m is partially ordered by a convex closed pointed cone C with a nonempty interior.
We assume that f is of class $C^{1,1}$ that is f is differentiable with f' Lipschitz continuous. The generalized Jacobian of the function f' is then called generalized Hessian of f and denoted by $\partial^2 f$.

<u>**Theorem 1.**</u> Assume that x_0 is a local weakly efficient solution of (VP). Then the following conditions hold

i) $f'(x_0)u \in (-\text{ int } C)^c$ for all $u \in R^n$
or equivalently there is $\xi \in C' \setminus \{0\}$ such that

$$\xi f'(x_0) = 0$$

ii) $\partial^2 f(x_0)(u,u) \cap (-\text{ int } C)^c \neq \emptyset$ for all $u \in R^n$ satisfying $f'(x_0)(u) \in -C \setminus$ int C, or equivalently for such u there exist $\eta \in C' \setminus \{0\}$ and $\varphi \in \partial^2 f(x_0)$ such that

$$\prec \eta, \varphi(u,u) \succ \geq 0 \ .$$

<u>Proof</u>. For i), suppose to the contrary that for some $u \in R^n$, one has $f'(x_0)(u) \in (-\text{int } C)^c$, i.e. $f'(x_0)(u) \in -\text{int } C$. Since

$$f'(x_0)(u) = \lim_{t \to 0} \frac{f(x_0 + tu) - f(x_0)}{t} \ ,$$

for $t > 0$ sufficiently close to 0 one has

$$\frac{f(x_0 + tu) - f(x_0)}{t} \in -\text{ int } C$$

which implies $f(x_0 + tu) - f(x_0) \in -$ int C, a contradiction with the fact that x_0 is locally weakly efficient.

For ii), suppose again to the contrary that there is some $u \in R^n$ such that

$$\begin{array}{ll} f'(x_0)(u) & \in -(C \setminus \text{ int } C) \\ \partial^2 f(x_0)(u,u) & \subset -\text{ int } C \ . \end{array}$$

Let V be a closed, convex neighborhood of $\partial^2 f(x_0)(u,u)$ such that $V \subseteq -$ int C. By the upper semicontinuity of generalized Hessian, there exists $\zeta > 0$ such that $\partial^2 f(x_0 + tu)(u,u) \subset V$ for every $t \in [0, \zeta]$.
This yields the inclusion

$$\text{cl conv}\{\partial^2 f(x)(u,u) : x \in [x_0, x_0 + \zeta u]\} \subset V \ .$$

By using Taylor's expansion (similar to the mean value theorem) we obtain

$$f(x_0+tu)-f(x_0)\in f'(x_0)(tu)+\text{ cl conv }\{\partial^2 f(x)(tu,tu):x\in[x_0,x_0+\zeta u]\}$$

$$\subseteq -t(C\setminus \text{ int } C)+t^2V\subset -\text{ int } C$$

for every $t\in(0,\zeta]$. This is a contradiction to the assumption of the theorem. ∎

Theorem 2. Assume that the following conditions hold at a point $x_0\in R^m$:

i) $\xi f'(x_0)=0$ for some $\xi\in \text{int } C'$;

ii) $\partial^2 f(x_0)(u,u)\subset(-C)^c$ for $u\in \text{Ker } f'(x_0), u\neq 0$.
Then x_0 is a local efficient solution of (VP).

Proof. If x_0 is not a local efficient solution of (VP), then there exists a sequence $\{x_i\}$ converging to x_0 such that

$$f(x_i)-f(x_0)\in -C\setminus\{0\},\ i=1,2,\cdots \tag{1}$$

without loss of generality we may assume that the sequence $\{u_i\}$ where $u_i=(x_i-x_0)\ /\ ||x_i-x_0||$ converges to some $u\in R^n$. Condition i) shows that $f'(x_0)(u)\notin -C\setminus\{0\}$. There are two possible cases : $f'(x_0)(u)\in(-C)^c$ and $f'(x_0)(u)=0$. The first case is impossible because (1) implies $f'(x_0)(u)\in -C$. Thus $u\in \text{Ker } f'(x_0)$. In view of ii) there exists a closed convex neighborhood V of $\partial^2 f(x_0)(u,u)$ in $(-C)^c$ such that $\partial^2 f(x)(v,v)\subset V$ whenever $||x-x_0||<\zeta$, $||v-u||<\zeta$ for some positive ζ small enough. By Taylor's expansion we obtain

$$f(x_i)-f(x_0)\in f'(x_0)(x_i-x_0)+\text{ cl conv}\{\partial^2 f(x)(x_i-x_0,x_i-x_0):x\in[x_0,x_i]\}$$

$$\subseteq ||x_i-x_0||\{f'(x_0)(u_i)+||x_i-x_0||\cdot \text{ cl conv }\{\partial^2 f(x)(u_i,u_i):x\in[x_0,x_i]\}\}\ .$$

Observe that $f'(x_0)(u_i)\subset(-C)^c\cup\{0\}$ by Condition i).
Moreover, for i sufficiently large, we have $||x_i-x_0||<\zeta$ and $||u_i-u||\leq\zeta$. Consequently, for such i, the above inclusions yield

$$\begin{aligned} f(x_i)-f(x_0) &\in ||x_i-x_0||\{(-C)^c\cup\{0\}+||x_i-x_0||V\} \\ &\subseteq (-C)^C\cup\{0\}+(-C)^c\subset(-C)^c \end{aligned}$$

which contradicts (1). The proof is complete. ∎

IV. SOLUTION METHODS

10. Two classical methods

Let us consider the following problem (VP) :

$$\underset{x \in X}{\text{Min}}\ f(x) = \big(f_1(x), \cdots, f_m(x)\big)$$

where X is a nonempty subset of R^n and the ordering cone of R^m is the positive orthant R^m_+.

a) Weighting method

This method consists of choosing weights $p_1, \cdots, p_m \geq 0$, not all zero and solving the associated scalar problem (P) by known techniques :

$$\underset{x \in X}{\text{Min}} \sum_{i=1}^{m} p_i f_i(x)\ . \qquad (P)$$

Theorem. For the problems (VP) and (P) above we have

i) If $p_i > 0, i = 1, \cdots, m$ then any optimal solution of (P) is an efficient solution of (VP).

ii) If $p_i \geq 0, i = 1, \cdots, m$ and not all are zero, then any optimal solution of (P) is a weakly efficient solution of (VP). If in addition that optimal solution is unique, then it is an efficient solution.

Proof. Observe that if $x_0 \in X$ is not an efficient solution of (VP), then there is $x \in X$ such that $f(x) \leq f(x_0), f(x) \neq f(x_0)$. Hence $\sum_{i=1}^{m} p_i f_i(x) < \sum_{i=1}^{m} p_i f_i(x_0)$ if all $p_i > 0$. This means that x_0 cannot be an optimal solution of (P). The case of weakly efficient solutions is proven in a similar way.
If in addition, x_0 is a unique solution of (P) [or more general, f (argmin (P)) is a singleton] where $p_i \geq 0$, $i = 1, \cdots, m$, not all zero, then for any other $x \in X$ with $f(x) \leq f(x_0)$, $f(x) \neq f(x_0)$ one has $\sum_{i=1}^{m} p_i f_i(x) \leq \sum_{i=1}^{m} p_i f_i(x_0)$ which implies that x solves (P). This contradicts the uniqueness assumption. The proof is complete. ∎

In practice, one chooses a family of weighting vectors $p = (p_1, \cdots, p_m)$ and solves the corresponding scalar problems (P). By this one may generate a subset of efficient solutions of (VP). In the case ii) of the theorem, in order to obtain an efficient solution, one proceeds as follows : let $p_1 > 0, \cdots, p_\ell > 0$ and $p_{\ell+1} = \cdots = p_m = 0$. One set $f_i^* = f_i(x_0)$ where x_0 is an optimal solution of (P). Then one solves a subsidary problem (P_*) :

$$\min \sum_{j=\ell+1}^{m} f_j(x)$$

$$x \in X, f_i(x) = f_i^* \quad i = 1, \cdots, \ell \ .$$

It is not difficult to see that any solution of (P_*) is an efficient solution of (VP).

b) Constraint Method.

In this method one minimizes one objective, while other objectives are considered as constraints.

Let us choose $k \in \{1, \cdots, m\}$, $L_j \in R$, $j = 1, \cdots, n$, $j \neq k$, and solve the scalar problem (P_k) :

$$\operatorname*{Min}_{x \in X} \ f_k(x)$$

$$f_j(x) \leq L_j \, , \ j = 1, \cdots, n \ , \ j \neq k \ .$$

Note that if L_j are small, then (P_k) may have no feasible solutions, if L_j are two big, then an optimal solution of (P_k) may be not efficient. We shall say that a constraint $f_j(x) \leq L_j$ is binding if every optimal solution of (P_k) verifies $f_j(x) = L_j$.

Theorem. Assume that x_0 is an optimal solution of (P_k) and all the constraints are binding. Then x_0 is an efficient solution of (VP).

Proof. If x_0 is not efficient, then there is some $x \in X$ such that $f_i(x) \leq f_i(x_0)$ for $i = 1, 2, \cdots, m$, $f(x) \neq f(x_0)$. It follows that x is a feasible solution of (P_k), and $f_k(x) = f_k(x_0)$. In other words x is an optimal solution of (P_k). Since the constraints are binding, we conclude $f_i(x) = f_i(x_0)$ for all $i = 1, \cdots, m$, a contradiction. ∎

Below is an algorithm to solve (VP).

Step 1 Solve

$$\min_{x \in X} f_i(x) \ .$$

Let $x^1, \cdots, x^m$ be optimal solutions.

Step 2 Construct the payoff table

$$\begin{array}{ccc} f_1(x^1) & \cdots & f_m(x^1) \\ \vdots & & \vdots \\ f_1(x^m) & \cdots & f_m(x^m) \\ M_1 & & M_m \\ m_1 & & m_m \end{array}$$

where

$$\begin{aligned} M_i &= \max\{f_i(x^1), \cdots, f_i(x^m)\} \\ m_i &= \min\{f_i(x^1), \cdots, f_i(x^m)\} \end{aligned}$$

Step 3 Choose $r = 1, 2, \cdots$ and solve (P_k) with

$$L_j = M_j - \frac{t}{r-1}(M_j - m_j) \ , \ t = 0, \cdots, r-1 \ .$$

If at a solution of (P_k), all the constraints are binding, then this solution is efficient. Otherwise, assuming $f_1, \cdots, f_\ell$ active, $f_{\ell+1}, \cdots, f_m$ $(\neq f_k)$ nonbinding, one solves (P_*) (in the previous method) to obtain an efficient solution. ∎

11. Normal Cones Method

This method is aimed at generating all efficient solutions of a linear multi-objective problem (VP) :

$$\text{Min } Cx$$

$$Ax \geq b$$

where C is an $m \times n-$matrix with m rows $C^1, \cdots, C^m$ and A is an $p \times n-$matrix with p rows $a^1, \cdots, a^p$, and $b \in R^p$.

Denote by $M := \{x : Ax \geq b\}$. We recall that the normal cone to M at $x_0 \in M$ is denoted by $N_M(x_0)$ and defined by

$$N_M(x_0) := \{v \in R^n :< v, x - x_0 > \leq 0 \ , \ x \in M\} \ .$$

Normal cone can be explicitely calculated by the following rule.

Lemma. Let $I(x_0)$ be the active index set at $x_0 \in M$, i.e.

$$I(x_0) = \{i \in \{1, \cdots, p\} : \prec a^i, x_0 \succ = b_i\}$$

and $\prec a^j, x_0 \succ > b_j$ if $j \notin I(x_0)$. Then $N_M(x_0) = \text{cone}\{-a^i : i \in I(x_0)\}$.

Proof. By a direct verification. ∎

Definition. Let $I \subseteq \{1, \cdots, p\}$. We say that I is normal if there is $x_0 \in M$ such that $N_M(x_0) = \text{cone}\{-a^i : i \in I\}$, and I is negative if cone $\{-a^i : i \in I\}$ contains a vector of the form $\sum_{i=1}^{m} \lambda_i C^i$ with $\lambda_1 > 0, \cdots, \lambda_m > 0$.

Let F be a face of the polyhedral convex set M. We say that F is an efficient solution face if every point of F is an efficient solution of (VP).

Theorem. Assume that there are no redundant constraints among $\prec a^i, x \succ b_i \ , \ i = 1, \cdots, p$. Let F be a face of M determined by the system

$$\prec a^i, x \succ = b_i \ , \ i \in I_F \subseteq \{1, \cdots, p\}$$

$$\prec a^j, x \succ \geq b_j \ , \ j \in \{1, \cdots, p\} \setminus I_F \ .$$

Then F is an efficient solution face if and only if I_F is negative and normal.

Proof. Invoke the lemma and use the fact that x_0 is an efficient solution of (VP) if and only if there exist $\lambda_1 > 0, \cdots, \lambda_m > 0$ such that

$$\prec \Sigma\lambda_i C^i, x - x_0 \succ \geq 0 \text{ for all } x \in M \ . \qquad \blacksquare$$

The next three prodedures allow to completely solve the problem (VP).

Procedure 1 (Finding an initial efficient solution vertex).

Step 1 Solve the system

$$\sum_{i=1}^{p} \mu_i a^i = \sum_{j=1}^{m} \lambda_j \cdot C^j \ , \ \mu_i \geq 0 \ , \ \lambda_j \geq 1 \ .$$

If it has no solutions, STOP ((VP) has no efficient solutions). Otherwise go to Step 2).

Step 2 Let λ be a solution of the above system. Put $v = C^T\lambda$. If $v = 0$, STOP (every feasible solution of (VP) is efficient). Otherwise solve the scalar linear problem

$$\min_{x \in M} \prec v, x \succ \ .$$

It is sure that this problem has optimal solutions. An optimal solution vertex of this problem is an efficient solution vertex of (VP).

Procedure 2 (Determining all efficient edges emanating from an initial efficient vertex x_0).

Step 1 Determine the active index set

$$I(x_0) := \{i \in \{1, \cdots, p\} :< a^i, x_0 >= b_i\} \ ,$$

and pick $I \subseteq I(x_0)$ with $|I| = n - 1$ not previously considered.
If rank $\{a^i : i \in I\} = n - 1$, go to Step 2.
Otherwise pick another $I \subseteq I(x_0)$.

Step 2 Verify whether I is negative by solving the system

$$\sum_{I \subset I} \mu_i a^i = \sum_{j=1}^{m} \lambda_j C^j \ , \ \mu_i \geq 0 \ , \ i \in I \ , \ \lambda_j \geq 1 \ , \ j = 1, \cdots, m \ .$$

If it has a solution, then go to Step 3 (I is negative).
Otherwise return to Step 1.

Step 3 Verify whether I is normal which implies that the edge determined by I is efficient.

Find $v \neq 0$ by solving

$$\prec a^i, v \succ = 0 \ , \ i \in I \ .$$

Solve the system

$$\prec a^i, x_0 + tv \succ \geq b_i \ , \ i = 1, \cdots, p \ .$$

Let the solution set be $[t_0, 0]$ or $[0, t_0]$ (t_0 may be ∞ or $-\infty$).

If $t_0 = 0$, then Return to Step 1 (I is not normal).

If $t_0 \neq 0$, then $[x_0, x_0 + t_0 v]$ is an efficient edge. Store it and return to Step 1 until no subset $I \subseteq I(x_0)$ with power $(n-1)$ left.

Procedure 3 (Finding an ℓ-dimensional efficient solution face adjacent to x_0).

Let $\{[x_0, x_0 + t_i v_i] \ ; \ i = 1, \cdots, k\}$ be the family of all efficient edges emanating from x_0 that have been obtained by Procedure 2 (assume $t_i > 0$).

Step 1 Pick $J \subseteq \{1, \cdots, k\}$ with $|J| = \ell$, not previously considered and set

$$x_J = \frac{x_0}{\ell + 1} + \sum_{j \in J} \lambda_j \ \frac{x_j}{\ell + 1}$$

where $x_j = x_0 + t_j v_j$ and $\lambda_j = t_j$ if t_j is finite, $\lambda_j = 1$ if $t_j = \infty$.

Step 2 Determine the active index set $I(x_J)$.

If $I(x_J) = \emptyset$, then Return to Step 1.

Otherwise go to Step 3.

Step 3 (Verify whether $I(x_J)$ is negative).

Solve the system of Step 2 (*Procedure 2*) with $I = I(x_J)$.

If it has a solution, go to Step 4 ($I(x_J)$ is negative).

Otherwise return to Step 1.

Step 4 (Find an ℓ-dimensional efficient face containing $[x_0, x_0 + t_j v_j] : j \in J]$.

Determine $J_0 := \{j \in \{1, \cdots, k\} : I_J \supseteq I(x_J)\}$.

Then the convex hull of $\{[x_0, x_0 + t_j v_j] : j \in J_0\}$ is an ℓ-dimensional efficient face adjacent to x_0.

Store it and pick J not containing J_0 with $|J| = \ell$ and continue Step 1.

Note that the set of efficient solutions of (VP) is pathwise connected, the above procedures allow to generate all efficient solutions of (VP) in a finite number of iterations. *Procedure 3* also gives a method generating all maximal efficient faces adjacent to a given efficient vertex.

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