ARE GENERALIZED DERIVATIVES USEFUL FOR GENERALIZED CONVEX FUNCTIONS ?

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Abstract. We present a review of some ad hoc subdifferentials which have been devised for the needs of generalized convexity such as the quasi-subdifferentials of Greenberg-Pierskalla, the tangential of Crouzeix, the lower subdifferential of Plastria, the infradifferential of Gutiérrez, the subdifferentials of Martinez-Legaz-Sach, Penot-Volle, Thach. We complete this list by some new proposals. We compare these specific subdifferentials to some all-purpose subdifferentials used in nonsmooth analysis. We give some hints about their uses. We also point out links with duality theories.

1 Introduction

The fields of generalized convexity and of nonsmooth analysis do not fit well with the image of mathematics as a well-ordered building. The notions are so abundant and sometimes so exotic that these two fields evoke the richness of a luxuriant nature rather than the purity of classical architecture (see for instance [25], [48], [117], [132], [187], and [169] with its rich bibliography for generalized convexity and [55], [69], [82], [205], [206], [148], [153] for nonsmooth analysis). Therefore, mixing both topics brings the risk of increasing the complexity of the picture.

However, we try to put some order and to delineate lines of thought around these two fields. It appears that in many cases the different concepts have comparable strengths. These comparisons enable one to derive a property in terms of a given subdifferential from the same property with a weaker notion of subdifferential by using sufficient conditions in order that both coincide for a function satisfying these conditions. Thus, the richness of the palette is rather an advantage. Moreover, to a certain extend, these specific subdifferentials can be treated in a somewhat unified way.

A first interplay between generalized convexity and nonsmooth analysis deals with generalized directional derivatives. Their comparison being reduced to inequalities, it suffices to give bounds for the derivatives one can use. Among the different concepts we present, we bring to the fore the incident derivative f^i (also called intermediate derivative, inner epi-derivative, upper epi-derivative, adjacent derivative). It is always a lower semicontinuous (l.s.c.) function of the direction; we show that it is also a quasiconvex function of the direction when the function is quasiconvex. This is an important feature of this derivative since Crouzeix has shown how to decompose such positively homogeneous functions into two convex parts. We present a variant of his decomposition which also preserves lower semicontinuity.

For what concerns subdifferentials, one already has the disposal of axiomatic approaches which capture the main properties of usual subdifferentials ([7], [85], [86], [149]...). These subdifferentials may be used for characterizing various generalized convexity properties. We recall such characterizations in sections 3 and 4, omiting important cases such as strong quasiconvexity ([45]), invexity ([33], [34], [71], [118], [219]-[222]...) log-concavity, rank-one convexity, rough convexity ([164]-[167]) and several others, but adding the case of paraconvexity.

Other subdifferentials exist which have been devised for the special needs of generalized convexity. We compare these specific notions (and add a few other variants) in section 5. We also look for links with all-purpose subdifferentials (section 6).

These comparisons are not just made for the sake of curiosity. As mentioned above, the relationships we exhibit enable one to deduce properties of a concept from known properties of another concept by using sufficient conditions to get equality or inclusion between the two subdifferentials. They also ease the choice of an appropriate subdifferential for a specific problem. In several instances the choice is dictated by the nature of the problem or by the duality theory which is avalaible. This fact is parallel to what occurs in nonsmooth analysis where the structure of the space in which the problem can be set influences the choice of the appropriate subdifferential.

We evoke shortly in section 8 the links with duality, leaving to other contributions the task of being more complete on this topic and on the other ones we tackle. Since subdifferentials would be of poor use if no calculus rule were avalaible, we give a short account of such rules in section 7 which seems to be missing in the literature, at least in a systematic way. We close the paper by presenting a new proposal of Martinez-Legaz and P.H. Sach [130] (and a variant of it) which has a special interest because it is small enough, close to usual subdifferentials and still adapted to generalized convexity.

Our study mainly focuses on subdifferentials, so that other tools of nonsmooth analysis such as tangent cones, normal cones, coderivatives, remain in the shadow. We also discard second order questions (see [11], [114], [183]...). It is probably regretful, but we tried to keep a reasonable size to our study. Still, since coderivatives are the adapted tool for studying multimappings (correspondences or relations) they certainly have a role to play in a field in which the sublevel set multimapping $r \mapsto F(r) := [f \leq r] := f^{-1}(]-\infty, r])$ associated with a function f plays a role which is more important than the role played by the epigraph of f. An explanation also lies in the freshness of the subject (see [85], [86], [137], [153] and their references for instance).

Applications to algorithms are not considered here; we refer to [16], [76], [77], [78], [170], [154], [194]-[196], [207] for some illustrations and numerous references. An application to well-posedness and conditioning will be treated elsewhere. For applications to mathematical economics we refer to [41], [48], [93], [129], [124], [131], [133], [187]...A nice application of lower subdifferentials to time optimal control problems is contained in [121], [119]. A recent interplay between quasiconvexity and Hamilton-Jacobi equations is revealed in [15], [217].

We hope that the reader will draw from the present study the conclusion that it is possible to stand outside the lost paradises of convexity and smoothness, and, hopefully, to go forward.

2 Generalized directional derivatives and their uses

A natural idea for generalizing convexity of differentiable functions expressed in terms of monotonicity of the derivative consists in replacing the derivative by a generalized derivative, so that nondifferentiable functions can be considered. A number of choices can be made. Let us recall some of them, the first one, the dag derivative, being a rather special notion introduced in [147] whose interest seems to be limited to the fact that it is the largest possible notion which can be used in this context. In the sequel f is an extended real-valued function on the n.v.s. X which is finite at some $x \in X$ and v is a fixed vector of X. The closed unit ball of X with center x and radius r is denoted by $B(x, r)$. The closure of a subset S of X is denoted by $\mathrm{cl}(S)$.

Thus the *dag derivative* of f is

$$f^{\dagger}(x, v) := \lim \sup_{(t,y) \to (0_+, x)} \frac{1}{t}(f(y + t(v + x - y)) - f(y)).$$

which majorizes both the *upper radial* (or upper Dini) derivative

$$f'_+(x, v) := \lim \sup_{t \to 0_+} \frac{1}{t}(f(x + tv) - f(x))$$

and the *Clarke-Rockafellar* derivative or circa-derivative

$$f^{\uparrow}(x, v) := \inf_{r>0} \lim \sup_{\substack{(t,y) \to (0_+, x) \\ f(y) \to f(x)}} \inf_{w \in B(v,r)} \frac{1}{t}(f(y + tw) - f(y)).$$

When f is Lipschitzian, $f^{\dagger}$ coincides with the *Clarke's derivative* f°:

$$f^{\circ}(x, v) := \lim \sup_{(t,y,w) \to (0_+, x, v)} \frac{1}{t}(f(y + tw) - f(y)).$$

The *Hadamard derivative* (or contingent derivative or lower epiderivative or lower Hadamard derivative)

$$f^{!}(x, v) := \lim \inf_{(t,u) \to (0_+, v)} \frac{1}{t}(f(x + tu) - f(x))$$

can also be denoted by $f'(x, v)$ in view of its importance. The *incident derivative* (or inner epiderivative)

$$f^{i}(x, v) := \sup_{r>0} \lim \sup_{t \searrow 0} \inf_{u \in B(v,r)} \frac{1}{t}(f(x + tu) - f(x))$$

is intermediate between the contingent derivative and the circa-derivative it is also bounded above by the upper Hadamard derivative (or upper hypo-derivative)

$$f^{\sharp}(x, v) := \lim \sup_{(t,w) \to (0_+, v)} \frac{1}{t}(f(x + tw) - f(x)) = -(-f)^{!}(x, v).$$

These derivatives can be ranked. Moreover, in the most useful cases such as one-variable functions, convex nondifferentiable functions, convex composite functions, finite maxima of functions of class C^1, these different notions coincide. Several of the preceding derivatives are such that their epigraphs are tangent cones (in a related sense) to the epigraph of the function. Unlike the convex case, such a geometrical interpretation does not bring much for generalized convex functions because their epigraphs are not as important as their sublevel sets.

The importance of the incident derivative stems from the following results: other derivatives share some of its properties such as lower semi-continuity (this is the case for the contingent and the circa-derivatives) or quasi-convexity (this is the case for the radial upper derivative, as shown in [39]) or accuracy, but not all. For instance Crouzeix has given in [38] an example of a quasiconvex function whose upper radial derivative is not l.s.c..

Proposition 1 *If f is quasiconvex and finite at x then the incident derivative $f^i(x,\cdot)$ is l.s.c. and quasiconvex.*

Proof. A simple direct proof can be given using the definitions: $f^i(x,v) \leq r$ iff for any sequence $(t_n) \to 0_+$ there exist sequences $(r_n) \to r$, $(v_n) \to v$ such that $f(x+t_n v_n) \leq f(x) + t_n r_n$; thus, if u, w are such that $f^i(x,u) \leq r$, $f^i(x,w) \leq r$, then any v in the interval $[u,w]$ satisfies $f^i(x,v) \leq r$.

A more elegant proof follows from the expression given in [212] Théorème 7 of the sublevel sets of the epi-limit superior q of a family (q_t) of functions on X parametrized by $t > 0$:

$$[q \leq r] = \bigcap_{s>r} \limsup_{t \to 0} [q_t \leq s].$$

This formula shows that q is quasiconvex whenever the functions q_t are quasiconvex. Since $q := f^i(x,\cdot)$ is the epi-limit superior of the family of quotients q_t given by $q_t(u) := t^{-1}(f(x+tu) - f(x))$ which are obviously quasiconvex, the result follows.□

Now we will make use of the following result which is a simple variant of results of Crouzeix ([37], [38], [39]); the relaxation of its assumptions will be useful.

Proposition 2 *Suppose h is a positively homogeneous quasiconvex extended real-valued function on X. Then each of the following two assumptions ensures that f is convex :*

(a) h is non negative ;

(b) there exists a nonempty dense subset D of the domain D_h of h on which h is negative.

Proof. Assertion (a) is proved in [39]. In order to prove assertion (b), using [39] Theorem 10 it suffices to show that for each $y \in X^*$ the Crouzeix function F given by

$$F(y,r) = \sup\{\langle y,x \rangle : x \in [h \leq r]\}$$

is concave in its second variable. It is obviously nondecreasing and since 0 belongs to the closure of D we have $F(y,0) \geq 0$, hence $F(y,1) \geq 0$. As h is positively homogeneous, $F(y,.)$ is also positively homogeneous. When $F(y,1) = 0$ we have $F(y,-1) \leq 0$ and $F(y,.)$ is concave. When $F(y,1) > 0$ we can find $x \in [h \leq 1]$ with $\langle y,x \rangle > 0$. Then there exists a sequence (x_n) in D with limit x ; we may suppose there exists $r > 0$ such that $\langle x_n, y \rangle > r$ for each n. Since $h(x_n) < 0$ we can find a sequence (t_n) of positive numbers with limit $+\infty$ such that $h(t_n x_n) \leq -1$ for each n. Then $F(y,-1) \geq \langle t_n x_n, y \rangle \to \infty$ and we get $F(y,.) \equiv +\infty$, a concave function. □

We will use jointly the preceding proposition and a decomposition of an arbitrary l.s.c. positively homogeneous function h which takes a special form when h is quasiconvex. Then, it differs from the Crouzeix's decomposition by the fact that its two terms are l.s.c. sublinear functions. Namely, let us set for an arbitrary l.s.c. positively homogeneous function h,

$$D = [h < 0], \qquad \overline{D} = clD,$$

$$h^{<}(x) = \begin{cases} h(x) & x \in \overline{D} \\ +\infty & x \in X \backslash \overline{D} \end{cases} \qquad h^{\geq}(x) = \begin{cases} 0 & x \in \overline{D} \\ h(x) & x \in X \backslash \overline{D}, \end{cases}$$

so that $\overline{D}$ replaces D in the Crouzeix's construction. Since h is l.s.c., $h(x) = 0$ for each $x \in \overline{D} \backslash D$, and $h^{\geq}$ coincides with the function h^{+} of the Crouzeix's decomposition, which is exactly the positive part of h. However $h^{<}$ differs from the corresponding term h^{-} of the Crouzeix's decomposition (which is not the negative part of h) on $\overline{D} \backslash D$ since $h^{-} \mid X \backslash D = \infty$ whereas $h^{<}(x) = 0$ on $\overline{D} \backslash D$ as observed above.

The proof of the following statement is immediate from what precedes since

$$\begin{array}{ll} [h^{<} \leq r] = [h \leq r] \text{ for } r < 0, & [h^{<} \leq r] = \overline{D} \text{ for } r \geq 0, \\ [h^{\geq} \leq r] = \emptyset \text{ for } r < 0, & [h^{\geq} \leq r] = [h^{<} \leq r] \text{ for } r \geq 0. \end{array}$$

Theorem 3 *For any positively homogeneous l.s.c. function h on X, the functions $h^{<}$ and $h^{\geq}$ are l.s.c.; they are convex when h is quasiconvex and*

$$h = \min\left(h^{<}, h^{\geq}\right).$$

We observe that when $h = f^{i}(x, .)$, the incident derivative at x, the set $\overline{D}$ is contained in the tangent set $T(S, x)$ to the sublevel $S := [f < f(x)]$ of f at x. In fact, for any $v \in D$ and any sequence $(t_n) \searrow 0$ there exists a sequence $(v_n) \to v$ with $\limsup_n t_n^{-1}(f(x + t_n v_n) - f(x)) < 0$, so that $x + t_n v_n \in S$ for n large enough : $v \in T(S, x)$; as $T(S, x)$ is closed we also have $\overline{D} \subset T(S, x)$.

We have proved the first part of the following statement. For the second part, we adapt the arguments of [39] Prop. 18 in order to identify $\overline{D}$.

Proposition 4 *When $h = f^{i}(x, .)$, the set $\overline{D}$ is contained in the tangent set $T(S, x)$. If f is quasiconvex, D is nonempty and if S is open (in particular if f is upper semicontinuous on S) one has $]0, \infty[(S - x) \subset D$ and $\overline{D} = T(S, x)$.*

We observe that the assumption that D is nonempty cannot be replaced with the assumption that S is nonempty (consider the function f on $I\!R$ given by $f(r) = r^3$ and take $x = 0$).

Proof. As h is quasiconvex $T(S, x)$ is the closure of $]0, \infty[(S - x)$, and it remains to prove that $]0, \infty[(S - x)$ is contained in D or equivalently that $S - x \subset D$. We may suppose $x = 0$. Let $u \in S - x$ and let $w \in D$. Since S is open, we can find $s \in]0, 1[$ and $v \in S - x$ such that

$$u = (1 - s) w + sv.$$

Given $t > 0$, let $p(t) := (1 - st)^{-1}(1 - s)t$ and let $(w_t) \to w$ be such that

$$\limsup_{t \to 0} \frac{1}{p(t)} (f(x + p(t) w_t) - f(x)) < 0.$$

Let us define u_t by

$$u_t := (1 - s) w_t + sv$$

so that $(u_t) \to u$. We can write

$$tu_t = (1 - st)\, p(t)\, w_t + stv$$

so that

$$f(x + tu_t) \leq \max\left(f(x + p(t)\, w_t),\, f(x + v)\right)$$

and

$$\frac{f(x + tu_t) - f(x)}{t} \leq \frac{f(x + p(t)\, w_t) - f(x)}{t}$$

since $t^{-1}(f(x + v) - f(x)) \to -\infty$ as $t \downarrow 0$. As $t^{-1}p(t) \to 1$ we get

$$\limsup_{t \searrow 0} \frac{f(x + tu_t) - f(x)}{t} \leq \limsup_{t \searrow 0} \frac{p(t)}{t} . \frac{1}{p(t)} \left(f(x + p(t)\, w_t) - f(x)\right) < 0.$$

Therefore $f^i(x, u) < 0$ and $u \in D$. □

3 Characterizations via directional derivatives

Let us recall some answers to the question: is it possible to characterize the various sorts of generalized convexity with the help of generalized derivatives? We limit our presentation to quasi-convexity and pseudo-convexity. We refer to [62], [63], [100], [103], [104], [147] for other cases, further details and proofs.

Theorem 5 *Let f be a l.s.c. function with domain C and let $f^?$be an arbitrary bifunction on $C \times X$. Among the following statements one has the implication $(b) \Rightarrow (a)$; when $f^? \leq f^\dagger$, or when f is continuous and $f^? \leq f^\circ$ one has $(a) \Rightarrow (b)$; when $f^? \leq f^\dagger$ and $f^?$ is l.s.c. in its second variable one has $(b) \Rightarrow (c)$; when for each $x \in X$ the function $f^?(x, \cdot)$ is positively homogeneous, l.s.c. and minorized by $f^!(x, \cdot)$ one has $(c) \Rightarrow (b)$.*

(a) f is quasiconvex ;

(b) f is $f^?$-quasiconvex i.e. satisfies the condition

$$(Q^?)\ (x, z) \in C \times X,\ f^?(x, z - x) > 0 \Rightarrow \forall y \in [x, z]\ \ f(z) \geq f(y);$$

(c) $f^?$ is quasimonotone, i.e. satisfies the relation

$$\min(f^?(x, y - x), f^?(y, x - y)) \leq 0 \text{ for any } x, y \in C.$$

The following definition generalizes a well known notion to non differentiable functions.

Definition 6 *The function f is said to be $f^?$-pseudoconvex if*

$$(P^?)\ \ x, y \in C,\ f^?(x, y - x) \geq 0 \Longrightarrow f(y) \geq f(x).$$

Theorem 7 *Let f be l.s.c. with $f^?$ l.s.c. in its second variable.*

(a) Suppose that for each local minimizer y of f one has $f^?(y, w) \geq 0$ for any $w \in X$ and suppose that $f^? \leq f^\dagger$. Then, if f is $f^?$-pseudoconvex, $f^?$ is pseudomonotone, i.e. for any $x, y \in C$ with $x \neq y$ one has $f^?(y, x - y) < 0$ whenever $f^?(x, y - x) > 0$.

(b) Conversely, suppose $f^?$ is pseudomonotone, sublinear in its second variable with $f^? \geq f^!$. Then, if f is continuous, it is pseudoconvex.

4 Subdifferential characterizations

Let us now consider the use of another generalization of derivatives, namely subdifferentials. We first consider subdifferentials which have sense for any type of function.

A well-known procedure to construct subdifferentials is as follows. With any of the directional derivatives $f^?$ considered above one can associate a subdifferential $\partial^?$ given by

$$\partial^? f(x) := \left\{ y \in X^* : \forall v \in X \ \langle y, v \rangle \leq f^?(x, v) \right\}.$$

For instance the Hadamard or *contingent (*resp. *incident) subdifferential* $\partial^! f(x)$ (resp. $\partial^i f(x)$) is the set of continuous linear forms minorizing $f^!(x, \cdot)$ (resp. $f^i(x, \cdot)$). Moreover

$$y \in \partial^! f\left(x\right) \Leftrightarrow \exists \varepsilon \in H : \ f\left(x + tv\right) \geq f(x) + \langle x^*, tv \rangle - \varepsilon\left(t, v\right) t.$$

where H denotes the set of $\varepsilon : I\!\!R \times X \to I\!\!R$ such that for any $u \in X$ one has $\lim_{(t,v) \to (0,u)} \varepsilon(t, v) = 0$. Several other notions which can be associated with a notion of tangent cone such as the circa-subdifferential of Clarke [28], the moderate subdifferential [134] and the concepts given in [157] and [204] can be constructed in such a way.

However several important classes of subdifferentials cannot be defined in the preceding way. This is the case for the approximate subdifferential ([81]- [85]), the limiting subdifferential ([135]-[137]), the proximal subdifferential, the viscosity subdifferential ([24]) and the *firm* (or Fréchet) *subdifferential.* The last one is given by

$$y \in \partial^- f\left(x\right) \Leftrightarrow f\left(x + u\right) \geq f\left(x\right) + \langle y, u \rangle - \varepsilon\left(u\right) \|u\| \ \text{with} \ \lim_{u \to 0} \varepsilon(u) = 0$$

and coincides with the contingent (or Hadamard) subdifferential in finite dimensional spaces.

Several authors have found convenient to formulate lists of axioms which enable one to consider various concepts in an unified way without bothering about the specific constructions ([7], [68], [85], [86], [149]...). A possible list is the following one. Here a subdifferential $\partial^?$ is a relation which associate with each f in a class $\mathcal{F}(X)$ of extended real-valued functions on a space X and each x in the domain $D(f) := dom f$ of f a subset $\partial^? f(x)$ of the dual space X^*.

Axioms for a subdifferential $\partial^?$

(S_1) If f and g coincide on a neighborhood of x then $\partial^? f(x) = \partial^? g(x)$;

(S_2) If f is convex then $\partial^? f(x)$ is the Fenchel subdifferential ;

(S_3) If f attains at x a local minimum then $0 \in \partial^? f(x)$;

(S_4) For any $f \in \mathcal{F}(X)$ and any $l \in X^*$ one has $\partial^?(f + l)(x) = \partial^? f(x) + l$.

Among the additional properties which may be satisfied is the following one.

Reliability (fuzzy principle [79], [80], [86], [149]**)** A triple $(X, \mathcal{F}(X), \partial^?)$ is said to be reliable if for any l.s.c. function $f \in \mathcal{F}(X)$, for any convex Lipschitzian function g, for any $x \in D(f)$ at which $f + g$ attains its infimum and for any $\varepsilon > 0$ one has

$$0 \in \partial f(u) + \partial g(v) + \varepsilon B^*,$$

for some $u, v \in B(x, \varepsilon)$ such that $\mid f(u) - f(x) \mid < \varepsilon$.

In particular, if $\mathcal{F}(X)$ is the class of l.s.c. functions on X and if $\partial^?$ is a given subdifferential, we say X is a $\partial^?$-reliable space. The preceding concept, which is a variant of the notion of trustworthiness introduced by A. Ioffe enables one to obtain a form of the Mean Value Theorem ; several other forms exist (see [7], [9], [92], [106], [111], [143] and their references).

Theorem 8 *(***Mean Value Theorem***). Let $(X, \mathcal{F}(X), \partial^?)$ be a reliable triple and let $f : X \to I\!R \cup \{\infty\}$ be a l.s.c. element of $\mathcal{F}(X)$, finite at $a, b \in X$. Then there exists $c \in [a, b[$ and sequences $(c_n), (c_n^*)$ such that $(c_n) \to c$, $(f(c_n)) \to f(c)$, $c_n^* \in \partial^? f(c_n)$ for each n and*

$$\liminf_n \langle c_n^*, b - a \rangle \geq f(b) - f(a),$$
$$\liminf_n \langle c_n^*, \tfrac{\|b-a\|}{\|b-c\|}(b - c_n) \rangle \geq f(b) - f(a).$$

Using this result (or other forms), characterizations of generalized monotonicity properties can be given. Let us first recall the most important generalized monotonicity properties which have been studied for multimappings.

Quasi-monotonicity A multimapping $F : X \rightrightarrows X^*$ is quasi-monotone if for any $x, y \in X$, $x^* \in F(x)$, $y^* \in F(y)$ one has

$$\max\{\langle x^*, x - y \rangle, \langle y^*, y - x \rangle\} \geq 0$$

Pseudo-monotonicity A multimapping $F : X \rightrightarrows X^*$ is pseudo-monotone if for any $x, y \in X$, one has the implication

$$\exists x^* \in F(x) : \langle x^*, y - x \rangle > 0 \Longrightarrow \forall y^* \in F(y) \quad \langle y^*, y - x \rangle > 0.$$

The following result has been proved under various degrees of generality by a number of authors. The one we present here is taken from [156]; it makes use of the subdifferential $\partial^\dagger$ deduced from the dag derivative $f^\dagger$. Incidentally, let us note that Proposition 2.3 of [9] brings closer the approach of [7], [8] and the present approach. This type of result is a natural generalization of a characterization of convexity by a monotonicity property of the derivatives.

Theorem 9 *Let X be a $\partial^?$-reliable space and let $f \in \mathcal{F}(X)$ be a l.s.c. function. Then $\partial^? f$ is monotone iff f is convex.*

The characterization of quasiconvexity which follows uses a condition, condition (b), which has been introduced by Aussel ([5], [6]) and which is obviously stronger than the condition:

$$(b') \text{ if } \langle x^*, y - x \rangle > 0 \text{ for some } x^* \in \partial^? f(x) \text{ then } f(y) \geq f(x).$$

Theorem 10 *Let $f : X \to I\!R \cup \{\infty\}$ be a l.s.c. function. Then, among the following conditions, one has the implications:*

(a) $\Longrightarrow$ (b) when $\partial^? \subset \partial^\dagger$ or when $\partial^? \subset \partial^\circ$ and f is continuous;
(b) $\Longrightarrow$ (c) when $\partial^? \subset \partial^\dagger$;
(c) $\Longrightarrow$ (a) when X is $\partial^?$-reliable.

(a) f is quasiconvex;
(b) if $\langle x^*, z - x \rangle > 0$ for some $x^* \in \partial^? f(x)$ then $f(z) \geq f(y)$ for each $y \in [x, z]$;
(c) $\partial^? f$ is quasimonotone.

The following generalization of pseudo-convexity can also be characterized with the help of subdifferentials.

Definition 11 *The function $f : X \to \overline{I\!R}$ is said to be $\partial^?$-pseudo-convex if*

$$\forall x, y \in X : f(y) < f(x) \quad \Longrightarrow \quad \forall x^* \in \partial^? f(x) : \langle x^*, y - x \rangle < 0.$$

The following characterization is similar to the one in [154] but it applies to a general class of subdifferentials. Hereafter we say that f is *radially continuous* (resp. u.s.c., resp. l.s.c.) if its restrictions to lines are continuous (resp. u.s.c., resp. l.s.c.).

Theorem 12 *Suppose X is $\partial^?$ reliable and $f \in \mathcal{F}(X)$ is l.s.c. and radially continuous. If $\partial^? f$ is pseudo-monotone then f is pseudoconvex for $\partial^?$. Conversely, if $\partial^? \subset \partial^\dagger$ and if f is pseudoconvex for $\partial^?$ then $\partial^? f$ is pseudo-monotone.*

Proof. The proof of the first assertion is similar to the proof of the corresponding assertion in [154] Theorem 4.1. Suppose f is pseudoconvex for $\partial^?$. Consider $x, y \in X$ such that

$$\begin{aligned} \exists x^* &\in \partial^? f(x) : \quad \langle x^*, y - x \rangle > 0, & (1) \\ \exists y^* &\in \partial^? f(y) : \quad \langle y^*, y - x \rangle \leq 0. & (2) \end{aligned}$$

and let us find a contradiction. We can find a neighborhood V of y such that

$$\langle x^*, y' - x \rangle > 0 \text{ for each } y' \in V.$$

By pseudoconvexity of f we have

$$\begin{aligned} f(y') &\geq f(x), \text{ for each } y' \in V, \\ f(x) &\geq f(y). \end{aligned}$$

Thus y is a local minimizer of f and by (S_3) $0 \in \partial^? f(y)$. Now let us show that this inclusion is impossible. In fact, as $x^* \in \partial^? f(x) \subset \partial^\dagger f(x)$ and as $\langle x^*, y - x \rangle > 0$ we have $f^\dagger(x, y - x) > 0$, hence

$$\limsup_n t_n^{-1}(f(x_n + t_n(y - x_n)) - f(x_n)) > 0$$

for some sequences $(t_n) \to 0_+$, $(x_n) \to x$. For n large enough we get

$$f(x_n + t_n(y - x_n)) > f(x_n)$$

and, as f is quasiconvex by [154] Corollary 3.1, we get $f(y) > f(x_n)$. Thus, by pseudo-convexity we obtain $0 \notin \partial^? f(y)$ and we get a contradiction.□

As an application of what precedes let us give a criterion for paraconvexity which removes the assumption of tangential convexity used in [156] (A. Jourani has informed us that he also has a criterion for paraconvexity [94] but we are not aware of the assumptions and conclusions of his results). Given a continuous convex function $h : X \to I\!R$ one says that a function f on X is *h-paraconvex* (or paraconvex if there is no danger of confusion, in particular if $h(x) = \frac{1}{2}\|x\|^2$, [211], [139], [161], [51], [52]...) if there exists $r > 0$ such that $f + rh$ is convex. To be more precise, one also says that f is paraconvex with index (at most) r. We need a property which is still closer to the notion of trustworthiness than the notion of reliability. Following [145] Definition 2, we say that X is *C-dependable* for $\partial^?$ if for any l.s.c. function f on X, for any convex Lipschitzian function g on X and for any $x^* \in \partial^?(f+g)(x)$, $\varepsilon > 0$ there exists $u, v \in B(x, \varepsilon)$, $u^* \in \partial^? f(u)$, $v^* \in \partial g(v)$ such that $\mid f(u) - f(x) \mid < \varepsilon$, $\|u^* + v^* - x^*\| < \varepsilon$.

Proposition 13 *Suppose X is C-dependable for $\partial^?$. If for any $x_i \in X$, $y_i \in \partial^? f(x_i)$, $z_i \in \partial h(x_i)$ for $i = 1, 2$ one has*

$$\langle y_1 - y_2, x_1 - x_2 \rangle \geq -r\langle z_1 - z_2, x_1 - x_2 \rangle$$

then f is h-paraconvex with index at most r. The converse holds when $\partial^?(-h) \subset -\partial h$, in particular when $\partial^?$ is the Fréchet or the Hadamard subdifferential.

Proof. Let $\overline{\partial}^?$ be the stabilized (or limiting) subdifferential associated with $\partial^?$: for x in the domain of f the set $\overline{\partial}^? f(x)$ is the set of weak* limit points of bounded nets $(y_i)_{i \in I}$ such that for some net $(x_i)_{i \in I} \to x$ with $(f(x_i))_{i \in I} \to f(x)$ one has $y_i \in \partial^? f(x_i)$ for each $i \in I$. Then the same inequality holds for y_1 and y_2 in $\overline{\partial}^? f(x_1)$ and $\overline{\partial}^? f(x_2)$ respectively. Moreover, setting $g := f + rh$, we have $\overline{\partial}^? g(x) \subset \overline{\partial}^? f(x) + r\partial h(x)$, as easily seen. Therefore $\overline{\partial}^? g$ is monotone and g is convex.

Conversely, if g is convex then the monotonicity of $\overline{\partial}^? g$ and the inclusions

$$\begin{aligned} \partial^? f(x) &\subset \overline{\partial}^? g(x) + r\overline{\partial}^?(-h)(x) \\ &\subset \overline{\partial}^? g(x) - r\partial h(x) \end{aligned}$$

(since ∂h coincides with its stabilization) entail the inequality of the statement.$\square$

In turn, generalized monotonicity properties can be characterized by generalized differentiability tools; we refer to [43] and [114] for two recent contributions to this question which is outside the scope of the present paper.

5 Specific subdifferentials and their relationships

It is likely that a specific tool will be more efficient than an all-purpose tool. In the case of generalized convexity, a number of subdifferentials have been introduced which are adapted to generalizations of convexity, especially for duality questions. Let us first review these notions by giving a list of some of the most important classical definitions and of some new variants. Later on we will consider some other approaches and explain the origin of several of these subdifferentials by their links with duality theories. We do not consider here localized versions of these subdifferentials, although they are likely to have better relationships with the general subdifferentials we considered above, since these subdifferential are of a local nature (see [123] as an instance of such a local subdifferential). It seems to us that the most important concepts are the Greenberg-Pierskalla's subdifferential (and its variants) because it is a general notion which is easy to handle and the Plastria's subdifferential because it is rather close to the usual subdifferential of convex analysis. However, for some

special situations, some other concepts are more adapted. We will see that most of these subdifferentials share similar calculus rules.
The **Greenberg-Pierskalla**'s subdifferential ([66]) :

$$y \in \partial^* f(x) \text{ iff } \langle u - x, y \rangle \geq 0 \Rightarrow f(u) \geq f(x)$$

A variant of it, the **star subdifferential** :

$$y \in \partial^\star f(x) \text{ iff } y \neq 0, \ \langle u - x, y \rangle > 0 \Rightarrow f(u) \geq f(x)$$

when x is not a minimizer of f and $\partial^\star f(x) = X^*$ when x is a minimizer of f.
The **Crouzeix's tangential** ([37])

$$y \in \partial^T f(x) \text{ iff } \forall\, r < f(x) \quad \sup_{u \in [f \leq r]} \langle u, y \rangle < \langle x, y \rangle$$

A variant of it, the **τ-tangential**,

$$y \in \partial^\tau f(x) \text{ iff } \forall\, r < f(x) \quad \sup_{u \in [f \leq r]} \langle u - x, y \rangle \leq r - f(x)$$

The **Plastria 's lower subdifferential** ([170])

$$y \in \partial^< f(x) \text{ iff } \forall u \in [f < f(x)] \ \langle u - x, y \rangle \leq f(u) - f(x)$$

A variant of it, the **Gutiérrez's infradifferential** ([67]),

$$y \in \partial^\leq f(x) \text{ iff } \forall u \in [f \leq f(x)] \ \langle u - x, y \rangle \leq f(u) - f(x).$$

The **Atteia-Elqortobi's subdifferential** or **radiant** subdifferential ([5], [1], [152])

$$y \in \partial^o f(x) \text{ iff } \langle x, y \rangle > 1 \text{ and } \forall u \in [y > 1] \ f(u) \geq f(x).$$

Variants of it, the **evenly radiant subdifferential** ([152], [151]):

$$y \in \partial^\wedge f(x) \text{ iff } \langle x, y \rangle \geq 1 \text{ and } \forall u \in [y \geq 1] \ f(u) \geq f(x),$$

and the **Thach's subdifferential** ([197], [198])

$$y \in \partial^H f(x) \text{ iff } \langle x, y \rangle = 1 \text{ and } \forall u \in [y \geq 1] \ f(u) \geq f(x).$$

The **shady subdifferential** ([152])

$$y \in \partial^\vee f(x) \text{ iff } \langle x, y \rangle < 1 \text{ and } \forall u \in [y < 1] \ f(u) \geq f(x).$$

A variant of it, the **evenly shady subdifferential** ([152])

$$y \in \partial^\nabla f(x) \text{ iff } \langle x, y \rangle \leq 1 \text{ and } \forall u \in [y \leq 1] \ f(u) \geq f(x).$$

We observe that in these notions the sublevels sets of the function play a key role, a natural fact for generalized convexity. For intance, for the variant of the Greenberg-Pierskalla's subdifferential we introduced we have, when $[f < f(x)]$ is nonempty,

$$y \in \partial^\star f(x) \Leftrightarrow y \in N([f < f(x)], x), \ y \neq 0,$$

where the (Fenchel) normal set to a subset S of X at some point $x \in X$ is given by

$$N(S,x) := \{y \in X^* : \forall u \in S \ \ \langle u - x, y \rangle \leq 0\}.$$

This observation leads us to introduce another variant of the Greenberg-Pierskalla's subdifferential by setting $\partial^{\nu} f(x) := N([f \leq f(x)], x)$ or, in other terms,

$$y \in \partial^{\nu} f(x) \Leftrightarrow (\langle u - x, y \rangle > 0 \Rightarrow f(u) > f(x)).$$

Continuity properties of this set are studied in [22]. Also, defining the polar set C^0 of a subset C of X by

$$C^o := \{y \in X^* : \forall u \in C \ \langle u, y \rangle \leq 1\},$$

and the half space $G(x) := [x > 1]$, we see that the radiant subdifferential satisfies

$$\partial^o f(x) = G(x) \cap [f < f(x)]^0.$$

Introducing the strict polar set $C^{\wedge}$ of C by

$$C^{\wedge} := \{y \in X^* : \forall u \in C \ \langle u, y \rangle < 1\}$$

and the antipolar sets $C^{\vee}$ and C^{∇} of C in which the relation $<$ is replaced by $>$ and $\geq$ respectively, we get analogous characterizations of $\partial^{\wedge} f(x)$, $\partial^{\vee} f(x)$ and $\partial^{\nabla} f(x)$, replacing $G(x)$ by appropriate half spaces.

It is natural to ask whether there are any relationships between the previous notions. The following ones are obvious. Here, as above, we denote by $\partial f(x)$ the Fenchel subdifferential defined by

$$y \in \partial f(x) \Leftrightarrow \forall u \in X \ \ f(u) - f(x) \geq \langle u - x, y \rangle$$

and for a subset C of X, 0^+C stands for the recession cone of C given by

$$0^+C := \{u \in X : \forall x \in X, \ \forall t \geq 0 \ \ x + tu \in C\}.$$

Proposition 14 *Let $H(x) := \{y : \langle x, y \rangle = 1\}$. For any extended real-valued function finite at x one has*

$$\begin{aligned} y &\in \partial^* f(x), \ \langle x, y \rangle > 0 \Rightarrow (\langle x, y \rangle)^{-1} y \in \partial^H f(x), \\ \partial^H f(x) &= \partial^{\wedge} f(x) \cap H(x) = \partial^{\nabla} f(x) \cap H(x) = \partial^* f(x) \cap H(x) \ . \end{aligned}$$

Proposition 15 *For any extended real-valued function finite at x one has*

$$\partial f(x) \subset \partial^{\leq} f(x) \subset \partial^{<} f(x) \subset \partial^{\tau} f(x) \subset \partial^T f(x) \subset \partial^* f(x) \subset \partial^{\star} f(x),$$

One may wonder whether the preceding subdifferentials are of the same nature. The answer is negative as $\partial^T f(x)$, $\partial^* f(x)$, $\partial^{\star} f(x)$, $\partial^{\nu} f(x)$ are cones whereas $\partial^{\leq} f(x)$, $\partial^{<} f(x)$, $\partial^{\tau} f(x)$ are just *shady* i.e. stable by homotheties of rate at least one. Thus, one is led to compare the first ones to the recession cones of the second ones or to the cones generated by the second ones.

Proposition 16 *Whenever $\partial^{<} f(x)$ is nonempty one has $\partial^{\star} f(x) = 0^+ \partial^{<} f(x)$.*

Proof. Given $y_0 \in \partial^< f(x)$, for any $y \in \partial^\star f(x)$ and any $t \geq 0$ we have $y_0 + ty \in \partial^< f(x)$, by the very definitions, hence $y \in 0^+\partial^< f(x)$. Conversely, if $y \in 0^+\partial^< f(x)$, for any $u \in [f < f(x)]$ and any $t > 0$ we have

$$\langle y_0 + ty, u - x \rangle \leq f(u) - f(x)$$

hence $\langle y, u - x \rangle \leq 0$ and $y \in \partial^\star f(x)$.□

Similarly, one has $\partial^\nu f(x) = 0^+\partial^\leq f(x)$ whenever $\partial^\leq f(x)$ is nonempty.

Let us compare more closely these various subdifferentials. For the variant we gave of the Greenberg-Pierskalla subdifferential there exists an easy criteria.

Proposition 17 *If f is radially u.s.c. at each point of $[f < f(x)]$ then $\partial^\star f(x) = \partial^* f(x)$. If there is no local minimizer of f in $f^{-1}(f(x))$ then*

$$\partial^\star f(x) = \partial^\nu f(x) \backslash \{0\} := N([f \leq f(x)], x) \backslash \{0\}.$$

Proof. For the first assertion, it suffices to prove that any $y \in \partial^\star f(x)$ belongs to $\partial^* f(x)$. When $y = 0$, both sets are equal to X^*. When $y \neq 0$, given $u \in X$ such that $\langle u - x, y \rangle \geq 0$ we cannot have $f(u) < f(x)$: taking v such that $\langle y, v \rangle > 0$ and setting $u_n := u + 2^{-n}v$ so that $f(u_n) < f(x)$ for n large enough since f is radially u.s.c. at x, we get a contradiction with $\langle u_n - x, y \rangle > 0$.

The second assertion follows from the fact that under its assumption the sublevel set $[f \leq f(x)]$ is contained in the closure of the strict level set $[f < f(x)]$, so that any $y \in \partial^\star f(x)$ is bounded above by $\langle y, x \rangle$ on this sublevel set : $y \in N([f \leq f(x)], x)$.□

As observed in [121], in general $\partial^< f(x)$ and $\partial^\leq f(x)$ are different (take $f = y^+$, the positive part of a non null continuous linear form and $x = 0$). The following statement gives a sufficient condition for their coincidence.

Proposition 18 *If there is no local minimizer of f on $f^{-1}(f(x))$ but x, in particular if $f(x) > \inf f(X)$ and if any local minimizer of f is a global minimizer or if f is semi-strictly quasiconvex, then $\partial^< f(x) = \partial^\leq f(x)$.*

Recall that f is said to be *semi-strictly quasiconvex* if $f((1-t)x_0 + tx_1) < f(x_0)$ whenever $t \in]0,1[$, and $f(x_1) < f(x_0)$; any convex function is obviously semi-strictly quasiconvex.

Proof. The first assertion can be proved as in [121] Cor. 4.22: given $y \in \partial^< f(x)$ and $u \in [f \leq f(u)]$ the inequality

$$\langle y, u - x \rangle \leq f(u) - f(x)$$

is obvious for $u = x$ and for $u \neq x$ follows from a passage to the limit when writing such an inequality with u replaced by u_n, where (u_n) is a sequence with limit u such that $f(u_n) < f(u)$ for each n; thus $y \in \partial^\leq f(x)$. Since any local minimizer of a semi-strictly quasiconvex function is a global minimizer, the last assertion is a consequence of the first one. □

The example of the one-variable function f given by $f(r) = r \wedge ((r-1) \vee 0)$, $x = 0$, shows that for a quasiconvex function the inclusions $\partial f(x) \subset \partial^\leq f(x) \subset \partial^< f(x)$ may be strict. However, for convex continuous functions one has a positive result which completes [67], [170] and [121] corol. 4.15.

Proposition 19 *Let f be a convex function finite at x. If x does not belong to the set M_f of minimizers of f then*

$$\partial^{<} f(x) = \partial^{\leq} f\,(x) = [1, \infty[\, \partial f\,(x)\,.$$

If moreover f is continuous at x then

$$\partial^{*} f\,(x) = \,]0, \infty[\, \partial f\,(x) =]0,1] \partial^{\leq} f(x) =]0,1] \partial^{<} f(x).$$

Proof. Since $\partial f\,(x) \subset \partial^{\leq} f\,(x) \subset \partial^{<} f(x)$ and since $\partial^{\leq} f\,(x)$ and $\partial^{<} f(x)$ are shady we have $[1, \infty[\, \partial f\,(x) \subset \partial^{\leq} f\,(x) \subset \partial^{<} f(x)$. Conversely, let $y \in \partial^{<} f\,(x)$. The very definition of $\partial^{<} f\,(x)$ ensures that x is a minimizer of the convex function $g : u \mapsto f(u) + \max(\langle x - u, y\rangle, 0)$. By standard subdifferential calculus rules we get some $r \in [0,1]$ such that $0 \in \partial f(x) - ry$. Since $0 \notin \partial f(x)$ we have $r \neq 0$. Thus $y \in [1, \infty[\, \partial f\,(x)\,.$

Now let $y \in \partial^{*} f\,(x)\,,$ so that x is a minimizer of f on $[-y \leq -\langle y, x\rangle]\,.$ Since $x \notin M_f$, one has $y \neq 0$ and the Slater's condition is satisfied. It follows that there exists $s \geq 0$ such that

$$0 \in \partial f\,(x) + s\,(-y)\,.$$

Again one has $0 \notin \partial f\,(x)\,,$ hence $s > 0$, and $y \in s^{-1} \partial f\,(x) : \partial^{*} f\,(x) \subset \,]0, \infty[\, \partial f\,(x)\,.$ Since $\partial^{*} f\,(x)$ is a cone and contains $\partial f\,(x)\,,$ the reverse inclusion holds. The last equalities are obtained by writing $y = tz$ with $t \in \,]0, \infty[$, $z \in \partial f\,(x)$ and considering separately the case $t < 1$ which follows from the inclusion $\partial f\,(x) \subset \partial^{\leq} f\,(x)$ and the case $t \geq 1$ for which $y \in \partial^{\leq} f(x)$.□

The example of the "canyon function" [121] f given by $f(r) = -(1 - r^2)^{1/2}$ for $r \in [-1, 1]$, $f(r) = 0$ for $\mid r \mid > 1$ shows that the equalities of the preceding proposition do not hold for a quasiconvex function.

For positively homogeneous functions easy comparisons are possible.

Proposition 20 *Let f be positively homogeneous with $f\,(0) = 0$. Then*

$$\partial^{T} f\,(0) = \mathbb{R}_{+} \partial^{<} f\,(0) = [0,1]\, \partial^{<} f\,(0)\,.$$

Proof. Since $\partial^{<} f\,(0)$ is shady the last equality holds and since $\partial^{T} f\,(0)$ is a cone containing $\partial^{<} f\,(0)\,,$ it suffices to prove that any $y \in \partial^{T} f\,(0)$ belongs to $\mathbb{R}_{+} \partial^{<} f\,(0)\,.$ By definition of $\partial^{T} f\,(0)$ we can find $s > 0$ such that $\langle u, y\rangle \leq -s$ for each $u \in [f \leq -1]\,.$ In particular we have

$$f\,(u) \geq \langle u, s^{-1} y\rangle$$

for each $u \in f^{-1}\,(-1)\,,$ and, by positive homogeneity, for each $u \in [f < 0]\,.$ Therefore $s^{-1} y \in \partial^{<} f\,(0)\,.$ □

In the following statement we say that f is *inf-Lipschitzian* if for each $r \in \mathbb{R}$ f is Lipschitzian on $[f \leq r]\,.$ The proposition generalizes results of Plastria [170] and Martinez-Legaz [121] Cor. 4.19 in which the Lipschitz assumption is either global or supplemented by a boundedness assumption (and X is finite dimensional). Let us observe that if f is convex continuous and if $S := [f \leq f(x)]$ is compact, then f is Lipschitzian on S. In fact, an elementary compactness argument shows that there exists $r > 0$ such that $S + rB_X \subset S' := [f \leq f(x) + 1]$ and as f is bounded below on S, f is Lipschitzian with rate $r^{-1}(f(x) + 1 - \inf_S f)$ on S.

Proposition 21 *Let f be radially continuous on X and Lipschitzian on the sublevel set $[f < f(x)]$. Then one has*

$$\partial^\star f(x) = \partial^* f(x) = \partial^T f(x) =]0, \infty[\partial^< f(x) =]0, 1]\partial^< f(x).$$

In particular, if f is radially continuous and is inf-Lipschitzian, the preceding equalities hold for any x in the domain of f.

Proof. We will prove that

$$\partial^* f(x) =]0, 1]\partial^< f(x).$$

In view of the equalities $\partial^\star f(x) = \partial^* f(x)$ by Proposition 17, $]0, 1]\partial^< f(x) =]0, \infty[\partial^< f(x)$, the string of equalities of the statement is a clear consequence of this relation. Thus it suffices to prove that any $y \in \partial^* f(x)$ belongs to $]0, 1]\partial^< f(x)$. When $y = 0$, we have $f(x) = \inf f(X)$, hence $\partial^< f(x) = X^*$ and as $\partial^< f(x) \subset \partial^T f(x) \subset \partial^* f(x)$ equality holds. Thus we may suppose $y \neq 0$. Let c be the Lipschitz rate of f on the sublevel set $[f < f(x)]$. Let $r < \|y\|, 0 < r < c$ and let $z := cr^{-1}y$. We can find $v \in X$ such that $\|v\| \leq r^{-1}$, $\langle v, y\rangle = 1$. Given $u \in [f < f(x)]$ we have $t := \langle u - x, y\rangle \leq 0$ and we can write

$$u - x = tv + w$$

for some $w \in X$ such that $\langle w, y\rangle = 0$. By definition of $\partial^* f(x)$ we have $f(x + w) \geq f(x)$. Since $f \mid [u, x + w]$ is continuous, there exists $s \in [0, 1]$ such that $f(u') = f(x)$ for $u' := (1 - s)u + s(w + x)$ and $f(u'') < f(x)$ for each $u'' \in [u, u']\backslash\{u'\}$. Then, the Lipschitz property we assume yields

$$\begin{aligned} f(u) - f(x) &= f(u) - f(u') \geq -c\|u - u'\| \\ &\geq -c\|u - (w + x)\| = -c\|tv\| \geq -cr^{-1} \mid t \mid = \langle u - x, z\rangle. \end{aligned}$$

Since u is arbitrary in $[f < f(x)]$ we get $y = c^{-1}rz \in]0, 1]\partial^< f(x)$. □

Let us end the present section with an application of the specific subdifferentials defined above to the directional derivatives of the function. We first not that the decomposition of $h = f^i(x, .)$ we defined above in Theorem 3 induces a decomposition of the incident subdifferential :

$$\partial^i f(x) = \partial h^<(0) \cap \partial h^\geq(0)$$

where

$$\partial^i f(x) := \left\{y \in X^* : y \leq f^i(x, \cdot)\right\}.$$

Moreover the Fenchel subdifferential of $h^< = f^i(x, .)^<$ at 0 coincides with the Plastria's subdifferential of h at 0 :

$$\partial h^<(0) = \partial^< h(0)$$

since by lower semicontinuity, $y \in \partial h^<(0)$ iff $y \mid D \leq h \mid D$, where $D = [h < 0]$ as $y \mid \overline{D} \leq 0$, $h \mid \overline{D}\backslash D \geq 0$.

On the other hand, we observe that we always have that $\partial h^\geq(0)$ is nonempty (it contains 0) and if X is a Banach space and if $f^i(x, .)$ does not take the value $+\infty$, $\partial h^\geq(0)$ is bounded.

In the following result we relate the Plastria's subdifferential of a function to the Plastria's subdifferential of its directional derivatives.

Proposition 22 *Let* $h = f^{?}(x,.)$ *for some l.s.c. directional derivative* $f^{?}$ *of* f *verifying* $f^{?}(x,.) \geq f^{!}(x,.)$, *the lower Hadamard derivative of* f *at* x. *Then if* $h = \min(h^{<}, h^{+})$ *is the decomposition of* h *according to Theorem 3 one has*

$$\partial^{<} f(x) \subset \partial^{<} h(0) = \partial h^{<}(0) \ \ (\supset \partial^{?} f(x)).$$

The reverse inclusion holds when f *is* $f^{?}$*-subconvex at* x *in the following sense:*

$$f(u) < f(x) \Longrightarrow f^{?}(x, u-x) \leq f(u) - f(x).$$

Note that if the restriction of f to $[f < f(x)] \cup \{x\}$ coincides with the restriction of a convex function then f is subconvex at x.

Proof. Let $y \in \partial^{<} f(x)$ and let $v \in D = [h < 0]$. Since $f^{!}(x,v) \leq h(v) < 0$, there exists sequences $(t_n) \downarrow 0$, $(v_n) \to v$ such that $t_n^{-1}(f(x + t_n v_n) - f(x)) \to r \leq h(v) < 0$. For n large enough we have $f(x + t_n v_n) < f(x)$ hence

$$\langle y, v_n \rangle \leq t_n^{-1}(f(x + t_n v_n) - f(x))$$

and passing to the limit we get

$$\langle y, v \rangle \leq h(v).$$

Therefore $y \in \partial^{<} h(0)$.

Now for each $y \in \partial^{<} h(0)$ we have $y \mid D \leq 0$ hence $y \mid clD \leq 0$ by continuity. Since $h \mid X \backslash D \geq 0$ we deduce from these two observations that $y \leq h^{<}$ on clD hence on X and $y \in \partial h^{<}(0)$. Conversely, given $y \in \partial h^{<}(0)$, for each $v \in D$ we have $\langle y, v \rangle \leq h^{<}(v) = h(v)$ hence $y \in \partial^{<} h(0)$.

The reverse inclusion is immediate when f is $f^{?}$-subconvex at x as for any $y \in \partial^{<} h(0)$ and any $u \in [f < f(x)]$ we have $h(u-x) < 0$ hence

$$\langle y, u-x \rangle \leq h(u-x) \leq f(u) - f(x). \square$$

A similar relationship holds for the Greenberg-Pierskalla's subdifferential $\partial^{*} f$ and its variant $\partial^{\star} f$.

Proposition 23 *Let* $y \in \partial^{\star} f(x)$ *(resp.* $y \in \partial^{*} f(x)$*). Then, for any directional derivative* $f^{?}$ *minorized by* $f^{!}$ *(resp.* f'_{-}*), one has the implication:*

$$f^{?}(x,v) < 0 \Rightarrow \langle y, v \rangle \leq 0 \text{ (resp. } \langle y, v \rangle < 0).$$

Conversely, if this implication holds and if f *is* $f^{?}$*-pseudoconvex, then* $y \in \partial^{\star} f(x)$ *(resp.* $y \in \partial^{*} f(x)$*).*

Proof. Let $y \in \partial^{\star} f(x)$ and let $f^{?} \geq f^{!}$. If $v \in X$ is such that $f^{?}(x,v) < 0$, then we also have $f^{!}(x,v) < 0$ and there exist sequences $(t_n) \to 0_{+}$, $(v_n) \to v$ such that

$$f(x + t_n v_n) - f(x) < 0,$$

so that x is not a minimizer and

$$\langle y, t_n v_n \rangle \leq 0.$$

Passing to the limit, we get $\langle y, v\rangle \leq 0$. If $y \in \partial^* f(x)$ and if $f^? \geq f'_-$ we can take $v_n = v$ for each n and we get $\langle y, t_n v\rangle < 0$, hence $\langle y, v\rangle < 0$.

Conversely suppose the implication holds and f is $f^?$-pseudoconvex. Then, given u such that $\langle u - x, y\rangle > 0$, we have $f^?(x, u - x) \geq 0$ and by $f^?$-pseudoconvexity we get $f(u) \geq f(x)$. The case of the Greenberg-Pierskalla's subdifferential is similar. We note that in both cases there is no restriction on the derivative for the converse.□

The implication above can be written

$$\partial^\star f(x) \subset \partial^\star f^?(x,\cdot)(0) \ \ (\text{resp. } \partial^* f(x) \subset \partial^* f^?(x,\cdot)(0))$$

and the converse means that the reverse inclusion holds.

6 Comparison with all-purpose subdifferentials

Now let us compare the specific subdifferentials we described with the elements of the family of all-purposes subdifferentials. Let us start with the case f is differentiable at x. As the subdifferentials $\partial^< f(x), ..., \partial^\star f(x)$ are shady, even when f is differentiable at x, we cannot expect that they coincide with the singleton $\{f'(x)\}$; they may at most coincide with $[1, \infty) f'(x)$. The following result is a step in this direction which generalizes [170] Prop. 4.14 and [121] Cor. 4.16 as here f is not supposed to be differentiable at x nor quasiconvex; moreover we get a result for a larger class of subdifferentials. As before, we suppose throughout that f is finite at x.

Proposition 24 *Suppose there exists some $y^\# \in -\partial^\# (-f)(x)$, $y^\# \neq 0$. Then for any $y \in \partial^\star f(x)$ (resp. $y \in \partial^< f(x)$) there exists some $r > 0$ (resp. $r \geq 1$) such that $y = ry^\#$. In particular, if f is Hadamard differentiable at x with $f'(x) \neq 0$, then, if $\partial^\star f(x)$ (resp. $\partial^< f(x)$) is nonempty,*

$$\partial^\star f(x) =]0, \infty[\, f'(x)$$

$$(\textit{resp. } \partial^< f(x) = [s, \infty[f'(x), \textit{ for some } s \geq 1).$$

Proof. Let $u \in X$ be such that $\langle y^\#, u\rangle > 0$. Then, as

$$(-f)^\# (x, -u) \geq \langle -y^\#, -u\rangle > 0$$

there exist sequences $(t_n) \searrow 0$, $(u_n) \to u$ such that $f(x - t_n u_n) < f(x)$ for each n. Given $y \in \partial^\star f(x)$, we have $y \neq 0$, since x is not a minimizer of f; it follows that $\langle y, -t_n u_n\rangle \leq 0$ and $\langle y, u\rangle \geq 0$. Therefore

$$\langle y^\#, u\rangle > 0 \Longrightarrow \langle y, u\rangle \geq 0.$$

The Farkas-Minkowski lemma ensures that there exists some $r \in I\!\!R_+$ such that $y = ry^\#$.

Now let us show that $r \geq 1$ when $y \in \partial^< f(x)$. Taking $u \in X$ such that $\langle y^\#, u\rangle > 0$, $\varepsilon > 0$ small enough, and sequences $(t_n) \downarrow 0$, $(u_n) \to u$ such that

$$t_n^{-1} ((-f)(x - t_n u_n) - (-f)(x)) \geq \langle -y^\#, -u\rangle - \varepsilon > 0$$

for each n, as $ry^{\#} = y \in \partial^{<} f(x)$, we get

$$\langle ry^{\#}, -t_n u\rangle \leq f(x - t_n u) - f(x) \leq \langle y^{\#}, -t_n u\rangle + t_n \varepsilon$$

hence $(r-1)\langle y^{\#}, u\rangle \geq -\varepsilon$ and $r \geq 1$, ε being arbitrary small.

Now let us suppose f is Hadamard differentiable with $f'(x) \neq 0$. Then x is not a minimizer of f, and $y^{\#} := f'(x) \in -\partial^{\#}(-f)(x)$ so that $\partial^{\star} f(x) \subset]0, \infty[f'(x)$. Since $\partial^{\star} f(x)$ is a cone we have the reverse inclusion when $\partial^{\star} f(x) \neq \emptyset$. Suppose now $\partial^{<} f(x)$ is nonempty and set

$$s = \inf \{r \geq 1 : rf'(x) \in \partial^{<} f(x)\}.$$

Since $\partial^{<} f(x)$ is closed, this infimum is attained. As $\partial^{<} f(x)$ is shady and closed , we get $[s, \infty[f'(x) \subset \partial^{<} f(x)$ and equality holds by what precedes. □

The following result weakens the convexity assumption on f made in Proposition 19 (for instance f may be nonconvex but Hadamard differentiable).

Proposition 25 *(a) Suppose f is finite at x and such that the contingent (resp. incident) derivative $f^{!}(x, .)$ (resp. $f^{i}(x, .)$) is convex and does not take the value -∞. If $\partial^{!} f(x)$ (resp. $\partial^{i} f(x)$) does not contain 0 then*

$$\partial^{\star} f(x) \subset cl\left(\mathbb{R}_{+} \partial^{!} f(x)\right)$$

$$\textit{(resp. } \partial^{\star} f(x) \subset cl\left(\mathbb{R}_{+} \partial^{i} f(x)\right) \textit{).}$$

If moreover $\partial^{!} f(x)$ (resp. $\partial^{i} f(x)$) is bounded, in particular when the derivative $f^{!}(x, .)$ (resp. $f^{i}(x, .)$) is continuous, then one can suppress the closure operation in the preceding inclusion.

(b) If a directional derivative $f^{?}(x, \cdot)$ is convex, finite-valued and minorized by the radial derivative $f'_{-}(x, .)$ or $f'_{+}(x, .)$ and if $0 \notin \partial^{?} f(x)$ then the associated subdifferential $\partial^{?} f$ satisfies

$$\partial^{*} f(x) \subset \mathbb{R}_{+} \partial^{?} f(x).$$

The second assertion applies to the Clarke's subdifferential or to the moderate subdifferential whenever it does not contain 0.

Proof. Let $y \in X^{*} \backslash cl\left(\mathbb{R}_{+} \partial^{!} f(x)\right)$. As $cl(\mathbb{R}_{+} \partial^{!} f(x))$ is convex and nonempty, the bipolar theorem yields some $u \in X$ such that $\langle y, u\rangle > 0$,

$$f^{!}(x, u) = \sup\left\{\langle z, u\rangle : z \in \partial^{!} f(x)\right\} \leq 0,$$

$f^{!}(x, .)$ being sublinear, l.s.c. and proper (as $f^{!}(x, 0) \leq 0$). Since $0 \notin \partial^{!} f(x)$ we can find some $w \in W$ such that $f^{!}(x, w) < 0$ and x is not a minimizer of f. Then, for $\varepsilon > 0$ small enough and for $v := u + \varepsilon w$, we have $\langle y, v\rangle > 0$ and $f^{!}(x, v) \leq f^{!}(x, u) + \varepsilon f^{!}(x, w) < 0$. However, the definition of $\partial^{\star} f(x)$ ensures that if $y \in \partial^{\star} f(x)$ then one has $f^{!}(x, v) \geq 0$ whenever $\langle y, v\rangle > 0$. The first inclusion follows and an analogous argument can be used with the incident derivative and the incident subdifferential.

When $\partial^{!} f(x)$ is bounded (in particular if $f^{!}(x, .)$ is continuous) and does not contain 0, the cone $\mathbb{R}_{+} \partial^{!} f(x)$ is closed.

It follows from the definition of $\partial^{*} f$ that, given $y \in \partial^{*} f(x)$, we have

$$f'_{-}(x, v) \geq 0 \text{ whenever } \langle y, v\rangle \geq 0,$$

hence $f^?(x,v) \geq 0$ whenever $\langle y,v\rangle \geq 0$ and $f^?(x,.) \geq f'_-(x,\cdot)$. The Hahn-Banach extension theorem yields some linear functional z such that $z \leq f^?(x,.)$ and $z \mid N = 0$ where $N = \mathrm{Ker}\, y$. Since $y \neq 0$ we get $z = \lambda\ y$ for some real number λ, and z is continuous. As $0 \notin \partial^? f(x)$ there exists some $v \in X$ such that

$$0 > f^?(x,v) \geq \langle z,v\rangle,$$

and by what precedes we have $\langle y,v\rangle < 0$. Therefore $\lambda > 0$ and $y = \lambda^{-1}z \in]0,\infty[\, \partial^? f(x) : \partial^* f(x) \subset]0,\infty[\, \partial^? f(x)$.□

The following converse is a variant of [131] Theorem 2.1 in which $\partial^?$ is the Fréchet subdifferential and f is supposed to be semistrictly quasiconvex.

Proposition 26 *If f is quasiconvex and u.s.c. on $[f < f(x)]$, then, for any subdifferential $\partial^?$ contained in $\partial^\sharp$ one has*

$$]0,\infty[\partial^? f(x) \subset \partial^* f(x) = \partial^\star f(x).$$

Proof. Let $y \in \partial^? f(x)$. Given $u \in X$ such that $\langle y, u-x\rangle > 0$ we have $f^\sharp(x, u-x) > 0$ so that there exist a sequence (t_n) in $]0,1[$ and a sequence $(u_n) \to u$ such that

$$f(x + t_n(u_n - x)) > f(x)$$

for each n. Since f is quasiconvex we get $f(u_n) > f(x)$ and by upper semicontinuity we obtain $f(u) \geq f(x)$. Thus $y \in \partial^\star f(x) = \partial^* f(x)$. Since these sets are closed cones the inclusion is proved.□

Let us observe that the inclusion of the preceding statement can be reformulated as: f is $\partial^?$-pseudo-convex.

7 Some properties of specific subdifferentials

Since the specific subdifferentials were often devised for quasiconvex functions, they are often insensitive to a scaling of the function in the sense that

$$\partial^?(rf)(x) = \partial^? f(x) \text{ for any } r > 0.$$

Such a choice stems from the fact that the data of level sets is the key information about the function. This viewpoint which is a sound viewpoint is not compatible with the usual approach which gives an importance to the values of the function and to its rate of change along directions. The preceding observation will be completed below by a result about rescaling.

Let us first observe that the specific subdifferentials we described above are not appropriate to characterizations of generalized convexity properties as they satisfy automatic generalized monotonicity properties.

Proposition 27 *For any function f and any subdifferential $\partial^?$ contained in ∂^ν the multifunction $\partial^? f$ is quasi-monotone.*

Proof. Let us show that if for any two pairs $(x_i, y_i) \in \partial^? f$, $i = 1,2$ the inequality

$$\max(\langle x_1 - x_2, y_1\rangle, \langle x_2 - x_1, y_2\rangle) < 0$$

is impossible. As $(x_1, y_1) \in \partial^? f \subset \partial^\nu f$ and $\langle x_2 - x_1, y_1\rangle > 0$ we have $f(x_2) > f(x_1)$ and similarly $f(x_1) > f(x_2)$, an impossibility.□

Proposition 28 *For any function f and any subdifferential $\partial^?$ contained in the infradifferential $\partial^{\leq}$ the multifunction $\partial^? f$ is pseudo-monotone.*

Proof. Let $(x_i, y_i) \in \partial^? f$, $i = 1, 2$ and let us show that

$$\langle x_2 - x_1, y_1 \rangle > 0 \Rightarrow \langle x_2 - x_1, y_2 \rangle > 0.$$

In fact, as $(x_1, y_1) \in \partial^{\leq} f$ the inequality $\langle x_2 - x_1, y_1 \rangle > 0$ implies $f(x_2) > f(x_1)$. Then, as $(x_2, y_2) \in \partial^{\leq} f$, we get $\langle x_1 - x_2, y_2 \rangle \leq f(x_1) - f(x_2) < 0$.□

Now one may ask whether these subdifferentials have interesting properties. We first note that they can be used to characterize minimizers.

Proposition 29 *Suppose f is finite at x. Then x is a minimizer of f on X iff 0 belongs to one of the subdifferentials between $\partial^{\leq} f(x)$ and $\partial^{\star} f(x)$ iff $\partial^{<} f(x) = ... = \partial^{\star} f(x) = X^*$.*

The proof is easy. The proof of the following observation is also immediate.

Proposition 30 *Suppose f is finite at x. Then the subdifferentials $\partial^{\nabla} f(x)$, $\partial^{\vee} f(x)$, $\partial^{\wedge} f(x)$, $\partial^{o} f(x)$, $\partial^{T} f(x)$, $\partial^{*} f(x)$ are convex and $\partial^{H} f(x)$, $\partial^{\leq} f(x)$, $\partial^{<} f(x)$, $\partial^{\tau} f(x)$, $\partial^{\star} f(x)$ are weak* closed and convex.*

Let us observe that unlike the case f is convex, for f quasiconvex, continuity of f around x does not entail nonemptiness of these subdifferentials.

Example. Let f be the one-variable function given by $f(r) = r^3$. Then $\partial^{<} f(x)$ is empty for $x = 0$.

However we have the following positive results. Recall that a set is evenly convex if it is the whole space or an intersection of open half spaces and that a function f is said to be evenly quasiconvex if its strict sublevel sets $[f < r]$, $r \in I\!R$, are evenly convex.

Proposition 31 *Let f be a quasiconvex function on X which is finite at x.*
(a) If f is evenly quasiconvex then $\partial^{*} f(x)$ is nonempty.
(b) If X is finite dimensional then $\partial^{\star} f(x)$ is nonempty.
(c) If X is an arbitrary n.v.s. and if f is u.s.c. on $[f < f(x)]$ then $\partial^{*} f(x)$ is nonempty.

Proof. We may suppose x is not a minimizer. In such a case, both assertions (b) and (c) follow readily from a separation theorem, $\{x\}$ and $[f < f(x)]$ being convex, nonempty and disjoint; in the second case the sublevel set is open. In case (a) the existence of a linear functional separating $\{x\}$ and $[f < f(x)]$ follows from the definition of an evenly convex set.□

The preceding result can be combined with comparison results.

Corollary 32 *If f is evenly quasiconvex, radially continuous, finite at x and Lipschitzian on $[f < f(x)]$, then $\partial^{<} f(x)$ and $\partial^{T} f(x)$ are nonempty.*

Proof. This is an immediate consequence of the preceding result and of Proposition 21.□

The star subdifferential and the Plastria's subdifferential satisfy a stability (or robustness, or closedness) property analogous to the one valid for the Fenchel subdifferential ∂.

Proposition 33 *Suppose f is finite at x, $(x_n) \to x$, $(f(x_n)) \to f(x)$ and $(y_n) \to y$ for the weak* topology with $y_n \in \partial^< f(x_n)$ (resp. $\partial^\star f(x_n)$) for each n. Then $y \in \partial^< f(x)$ (resp. $\partial^\star f(x)$).*

Proof. Let (x_n), (y_n) be as in the statement (bounded nets can also be used): $y_n \in \partial^< f(x_n)$ for each n, $(x_n) \to x$, $(f(x_n)) \to f(x)$ and $(y_n) \to y$. Then given $u \in [f < f(x)]$ we can find m such that $u \in [f < f(x_n)]$ for each $n \geq m$. It follows that

$$\langle u - x_n, y_n \rangle \leq f(u) - f(x_n).$$

Taking limits we get

$$\langle u - x, y \rangle \leq f(u) - f(x),$$

so that $y \in \partial^< f(x)$. When $y_n \in \partial^\star f(x_n)$ for each n we have $\langle u - x_n, y_n \rangle \leq 0$ for $n \geq m$ and we get $\langle u - x, y \rangle \leq 0$ so that $y \in \partial^\star f(x)$.□

Since the specific subdifferentials we described are intended to be applied to classes of generalized convex functions which are not stable under addition, the fact that there is no simple sum rule for these subdifferentials is not a serious drawback. A compensation lies in pleasant rules for operations which are important for those classes.

Proposition 34 *Suppose $f = g \circ A$ where $A : X \to W$ is a continuous linear map between the two n.v.s. X, W and $g : W \to \overline{I\!R}$ is finite at $w = g(x)$. Then, if $\partial^?$ is one of the subdifferentials $\partial^\nabla, \partial^\vee, \partial^H, \partial^\wedge, \partial^o, \partial^\leq, \partial^<, \partial^T, \partial^\tau, \partial^*, \partial^\star, \partial^\nu$ one has*

$$\partial^? g(w) \circ A \subset \partial^? f(x).$$

If A is surjective and open one has $\partial^\leq g(w) \circ A = \partial^\leq f(x)$ and $\partial^\nu g(w) \circ A = \partial^\nu f(x)$; if moreover f has no local minimizer on $f^{-1}(f(x))$ then $\partial^< g(w) \circ A = \partial^< f(x)$ and $\partial^\star g(w) \circ A = \partial^\star f(x)$.

Proof. The case of $\partial^<$ is treated in [170] Th. 3.5. Let us consider the case of $\partial^\star$. Given $z \in \partial^\star g(w)$, for any $u \in [f < f(x)]$ we have $v := A(u) \in [g < g(w)]$ hence $\langle z \circ A, u - x \rangle = \langle z, v - w \rangle \leq 0$ and $y := z \circ A \in \partial^\star f(x)$. The other cases are similar.

Conversely, suppose A is surjective and open and let $y \in \partial^\leq f(x)$. Given $v \in A^{-1}(0)$ and $\varepsilon \in \{-1, 1\}$ we have $f(x + \varepsilon v) = f(x)$ hence $\langle y, \varepsilon v \rangle \leq 0$ and $\langle y, v \rangle = 0$. Thus there exists a continuous linear form z on W such that $y = z \circ A$. Now for each $v \in [g \leq g(w)]$ and for each $u \in A^{-1}(v)$ one has $u \in [f \leq f(x)]$ hence

$$\langle z, v - w \rangle = \langle y, u - x \rangle \leq f(u) - f(x) = g(v) - g(w),$$

so that $z \in \partial^\leq g(w)$. When f has no local minimizer on $f^{-1}(f(x))$ and when $y \in \partial^\star f(x)$, given $v \in A^{-1}(0)$ and $\varepsilon \in \{-1, 1\}$ we can find sequences $(v_n^\varepsilon) \to v$ with $f(x + \varepsilon v_n^\varepsilon) < f(x)$ for each n, so that $\langle y, \varepsilon v_n^\varepsilon \rangle \leq 0$ and $\langle y, v \rangle = 0$. Taking z in the dual space of W such that $y = z \circ A$ we see that for each $v \in [g < g(w)]$ and for each $u \in A^{-1}(v)$ one has $u \in [f < f(x)]$ hence

$$\langle z, v - w \rangle = \langle y, u - x \rangle \leq 0,$$

and $z \in \partial^\star g(w)$. Moreover, in such a case we have $\partial^< g(w) \circ A \subset \partial^< f(x) = \partial^\leq f(x) = \partial^\leq g(w) \circ A$ and equality holds since $\partial^\leq g(w) \subset \partial^< g(w)$.□

Another important chain rule is the following.

Proposition 35 *Let $f = h \circ g$ where $g : X \to I\!R$ and $h : I\!R \to I\!R$ is nondecreasing. Then, if $\partial^?$ is one of the subdifferentials $\partial^\nabla, \partial^\vee, \partial^H, \partial^\wedge, \partial^o, \partial^<, \partial^*, \partial^\star$, one has*

$$\partial^? g(x) \subset \partial^? f(x).$$

If h is increasing these inclusions are equalities and the result also holds for $\partial^? = \partial^\leq, \partial^\nu$. If h is l.s.c. at $g(x)$ the inclusion $\partial^T g(x) \subset \partial^T f(x)$ also holds.

Proof. The inclusion is immediate since $[f < f(x)] \subset [g < g(x)]$. When h is increasing equality holds and we also have $[f \leq f(x)] = [g \leq g(x)]$ as the inequality $g(u) > g(x)$ implies $f(u) > f(x)$.

When h is l.s.c. at $g(x)$, given $r < f(x)$, we can find $s < g(x)$ such that $r < h(s)$. Then $[f \leq r] \subset [g \leq s]$ and the inclusion $\partial^T g(x) \subset \partial^T f(x)$ follows.□

Taking h given by $h(r) = r \wedge c := \min(r, c)$, we deduce that if $x \in f^{-1}(c)$ we have $\partial^?(f \wedge c)(x) \subset \partial^? f(x)$; in fact equality holds. Such a result can be deduced from the following rule.

Proposition 36 *Let $(f_i)_{i\in I}$ be an arbitrary family of functions finite at x and let $f := \inf_{i\in I} f_i$. Suppose $I(x) := \{i \in I : f_i(x) = f(x)\}$ is nonempty. Then, for $\partial^? = \partial^\nabla, \partial^\vee, \partial^H, \partial^\wedge, \partial^o, \partial^\leq, \partial^<, \partial^T, \partial^\tau, \partial^*, \partial^\star, \partial^\nu$ one has*

$$\partial^? f(x) \subset \bigcap_{i\in I(x)} \partial^? f_i(x).$$

If $I(x) = I$, then equality holds for $\partial^\nabla, \partial^\vee, \partial^H, \partial^\wedge, \partial^o, \partial^<, \partial^*, \partial^\star$; if moreover $I(u)$ is nonempty for each $u \in X$ (in particular if I is finite) then equality holds for $\partial^\leq, \partial^\nu$.

Proof. The first assertion follows from the definitions and the two inclusions $[f_i < f_i(x)] \subset [f < f(x)]$, $[f_i \leq f_i(x)] \subset [f \leq f(x)]$ for $i \in I(x)$. For the converse, one observes that if $I(x) = I$ and if $u \in [f < f(x)]$ (resp. $u \in [f \leq f(x)]$ and if $I(u)$ is nonempty), then, for some $i \in I$ one has $u \in [f_i < f_i(x)]$ (resp. $u \in [f_i \leq f_i(x)]$); the reverse inclusion then follows easily by taking an infimum over such i's. □

A similar result holds for suprema.

Proposition 37 *Let $(f_i)_{i\in I}$ be an arbitrary family of functions finite at x and let $f := \sup_{i\in I} f_i$. Suppose $I(x) := \{i \in I : f_i(x) = f(x)\}$ is nonempty. Then, for $\partial^? = \partial^\nabla, \partial^\vee, \partial^H, \partial^\wedge, \partial^o, \partial^\leq, \partial^<, \partial^T, \partial^\tau, \partial^*, \partial^\star, \partial^\nu$ one has*

$$\partial^? f(x) \supset co\left(\bigcup_{i\in I(x)} \partial^? f_i(x)\right).$$

Equality holds for ∂^ν if I is finite and equal to $I(x)$, if $C_i := [f_i \leq f_i(x)]$ is convex for each $i \in I(x)$ and if either for some $k \in I(x)$ one has $C_k \cap (\bigcap_{i\in I(x)\setminus\{k\}} \mathrm{int} C_i) \neq \emptyset$, or X is a Banach space, each C_i is closed and $X^I = \Delta - I\!R_+ \prod_{i\in I}(C_i - x)$, where Δ is the diagonal of X^I. If moreover each f_i is radially u.s.c. and has no local minimizer on $f_i^{-1}(x)$ then equality holds for ∂^* and $\partial^\star$. If furthermore each f_i is radially continuous on X and Lipschitzian on C_i then equality holds for $\partial^<$.

When $I(x)$ has two elements j, k only, the qualification condition of the preceding statement can be rewritten in the simpler form

$$X = I\!\!R_+(C_j - x) - I\!\!R_+(C_k - x).$$

Proof. Again the result follows from the definitions and the inclusions $[f < f(x)] \subset [f_i < f_i(x)]$, $[f \leq f(x)] \subset [f_i \leq f_i(x)]$ for $i \in I(x)$. When $\partial^? f(x)$ is closed and convex, one can replace co by $\overline{co}$. The second assertion is a consequence of Proposition 17 and of the calculus of normal cones since $[f \leq f(x)] = \bigcap_{i \in I}[f_i \leq f_i(x)]$ when $I = I(x)$ is finite. The last assertions follow from Proposition 17 and Proposition 21$\square$

The following result about performance functions is also very simple; it is important in connection with duality questions.

Proposition 38 *Let W and X be n.v.s. and let $f : W \times X \to \overline{I\!\!R}$,*

$$\begin{aligned} p(w) &:= \inf_{x \in X} f(w, x), \\ S(w) &:= \{x \in X : f(w, x) = p(w)\}. \end{aligned}$$

Then for $\partial^? = \partial^\nabla, \partial^\vee, \partial^H, \partial^\wedge, \partial^o, \partial^<, \partial^T, \partial^\tau, \partial^*, \partial^\star$, and for each $x \in S(w)$ one has $z \in \partial^? p(w)$ iff $(z, 0) \in \partial^? f(w, x)$.

Proof. The result is essentially a consequence of the fact that $v \in [p < p(w)]$ iff there exists $u \in X$ such that $(v, u) \in [f < f(w, x)]$ and of the relation

$$\langle (z, 0), (v - w, u - x) \rangle = \langle z, v - w \rangle.$$

For instance, if $z \in \partial^< p(w)$, then for each $(v, u) \in [f < f(w, x)]$ we have $p(v) \leq f(v, u) < f(w, x) = p(w)$ hence

$$\langle z, v - w \rangle \leq p(v) - p(w) \leq f(v, u) - f(w, x)$$

and $(z, 0) \in \partial^< f(w, x)$. Conversely, if this relation holds, then for each $v \in [p < p(w)]$ we can find some $u \in X$ such that $(v, u) \in [f < f(w, x)]$, so that

$$\langle z, v - w \rangle \leq \inf\{f(v, u) - f(w, x) : u \in [f(v, \cdot) < f(w, x)]\} = p(v) - p(w)$$

and $z \in \partial^< p(w)$.$\square$

The proof above shows that the implication $z \in \partial^\leq p(w) \Rightarrow (z, 0) \in \partial^\leq f(w, x)$ also holds and that it is an equivalence when for each $v \in W$ the set $S(v)$ of minimizers of $f(v, \cdot)$ is nonempty.

The preceding results can be used to compute the subdifferential of the level sum $s := g \overset{+}{\vee} h$ given by

$$s(w) := \inf_{x \in X} g(w - x) \overset{+}{\vee} h(x).$$

This operation, which is the analogue of the infimal convolution of convex analysis is of fundamental importance for quasiconvex analysis inasmuch as the usual sum does not preserve quasiconvexity whereas supremum does (see [12], [202], [203] ...). Moreover, one can check that the strict sublevel sets of s are given by

$$[s < r] = [g < r] + [h < r],$$

whereas the sublevel sets satisfy

$$[s \leq r] = [g \leq r] + [h \leq r]$$

whenever the infimum is attained in the formula defining s (then one says that the level sum is exact). The following rule, which mimics the classical rule for the infimal convolution, is a simple consequence of these formulas about sublevel sets.

Proposition 39 *Suppose the level sum $s := g \overset{+}{\vee} h$ of g and h is finite and exact at w. Then, for each $x \in X$ such that $s(w) = g(w-x) = h(x)$ one has $\partial^\nu s(w) = \partial^\nu g(w-x) \cap \partial^\nu h(x)$ and if x is such that $w - x$ (resp. x) is not a local minimizer of g (resp. h) one has $\partial^\star s(w) = \partial^\star g(w-x) \cap \partial^\star h(x)$. When g and h are radially u.s.c. a similar relation holds for the Greenberg-Pierskalla's subdifferential.*

Proof. The first assertion is obvious. Given $y \in \partial^\star s(w)$ and given $u \in [g < r]$, with $r = s(w) = g(w-x) = h(w)$ we take $x_n \in [h < r]$ with $(x_n) \to x$. Then $w_n := u + x_n \in [s < r]$, hence $\langle y, u + x_n - w \rangle \leq 0$. Taking limits it follows that $\langle y, u - (w - x) \rangle \leq 0$, hence $y \in \partial^\star g(w-x)$. The inclusion $y \in \partial^\star h(x)$ is similar. The reverse inclusion $\partial^\star s(w) \supset \partial^\star g(w-x) \cap \partial^\star h(x)$ is easy. The last assertion is a consequence of Proposition 17.□

Let us point out the interest of the specific concepts of subdifferential for optimization problems.

Proposition 40 *Suppose x is a minimizer of a quasiconvex function f on a convex subset C of X but x is not a local minimizer of f on X. Suppose X is finite dimensional (resp. f is u.s.c. on $[f < f(x)]$). Then there exists some $y \neq 0$ verifying $y \in \partial^\star f(x) \cap (-N(C,x))$ (resp. $y \in \partial^* f(x) \cap (-N(C,x))$).*

Proof. The sets $S = [f < f(x)]$ and C are convex, nonempty and disjoint, so that in both cases there exists $y \in X^* \backslash \{0\}$, $r \in \mathbb{R}$ such that

$$\langle y, w - x \rangle \geq r \geq \langle y, u - x \rangle \quad \forall w \in C, \ \forall u \in S.$$

Taking $w = x$ we get $r \leq 0$ so that $y \in \partial^\star f(x)$ (in fact $y \in \partial^* f(x)$ when S is open, and this occurs when f is u.s.c. on S). Using the assumption that there exist points $u \in S$ arbitrarily close to x we get $r = 0$, so that $-y \in N(C,x)$.□

A multiplier rule of the Karush-Kuhn-Tucker type can be deduced from the preceding proposition and from Proposition 17 (see also [121] Prop. 6.1 and numerous items of the bibliography [156]).

Corollary 41 *With the assumptions of the preceding proposition, suppose $C = g^{-1}(\mathbb{R}_-)$ where $g : X \to \mathbb{R}$ is such that $g(x) = 0$ and g has no local minimizer on $g^{-1}(0)$. Then, $\partial^\star f(x) \cap (-\partial^\star g(x)) \neq \{0\}$. If f and g are radially u.s.c. then $\partial^* f(x) \cap (-\partial^* g(x)) \neq \{0\}$. If moreover f and g are radially continuous and Lipschitzian on $[f < f(x)]$ and $[g < g(x)]$ respectively then there exists $\lambda > 0$ such that $0 \in \partial^< f(x) + \lambda \partial^< g(x)$.*

Proof. The first assertion stems from the relation $N(C,x) = \partial^\star g(x)$ when g has no local minimizers on $g^{-1}(0)$. The other ones are consequences of Propositions 17, 21.□

Corollary 42 *Suppose x is a solution to the problem*

$$\textit{minimize } f(u) \quad : \quad g_i(u) \leq 0 \ \ \forall i \in I,$$

where I is a finite set, f and g_i are continuous, quasiconvex, $g_i(x) = 0$ for each $i \in I$, f (resp. g_i) is Lipschitzian on $[f < f(x)]$ (resp. on $g_i^{-1}(\mathbb{R}_-)$). Suppose there exists some $x_o \in X$ such that $g_i(x_o) < 0$ for each $i \in I$. Then there exist $y \in \partial^< f(x)$, $y_i \in \partial^< g_i(x)$, $\lambda_i \in \mathbb{R}_+$ such that

$$y + \sum_{i \in I} \lambda_i y_i = 0.$$

Proof. When x is a minimizer of f on X, we can take arbitrary $y_i \in \partial^< g_i(x)$, $\lambda_i \in \mathbb{R}_+$. When x is not a minimizer of f on X, we set $g = \max_i g_i$, and we apply the preceding corollary and 37. □

8 Subdifferentials obtained by duality schemes

The nature of the preceding subdifferentials is enlightened if one considers them as associated with duality schemes, following ideas of Moreau [138], Balder [14], Dolecki-Kurcyusz [50], Penot-Volle [158], [159], Martinez-Legaz [120], [121], [122], [125], Pallaschke-Rolewicz [139] and many others ([60], [87], [108], [140], [189]...).

In fact, a subdifferential can be associated with any conjugacy. Let us recall this simple process. Given two sets X, Y and a function $c : X \times Y \to \overline{\mathbb{R}}$ called a coupling function or a pairing, one may define a conjugacy by setting for any extended real-valued function f on X

$$f^c(y) := -\inf_{x \in X}(f(x) - c(x, y)).$$

Then one can define for f finite at x

$$y \in \partial^c f(x) \Leftrightarrow f^c(y) + f(x) = c(x, y) \in \mathbb{R}.$$

Equivalently

$$y \in \partial^c f(x) \Leftrightarrow c(x, y) \in \mathbb{R}, \ \forall u \in X \ f(u) \geq f(x) + c(u, y) - c(x, y).$$

Introducing the biconjugate of f by $f^{cc} := (f^c)^c$ one can prove easily the following result.

Proposition 43 *If f is finite at x and if $\partial^c f(x)$ is nonempty, then $f^{cc}(x) = f(x)$.*

It is shown in [121], [158], [159] that a number of conjugacies for quasiconvex functions on a n.v.s. X can be derived from the evaluation mapping $c : X \times Y_K \to \overline{\mathbb{R}}$ where $Y_K := K \circ X^*$ is the set of functions on X which are obtained as $k \circ y$ with $y \in X^*$ and k belongs to an appropriate class K of (usually nondecreasing) extended real-valued one-variable functions. Note that the class K is often parametrized by the set of real numbers, so that the conjugate can be considered as defined on $X^* \times \mathbb{R}$. This fact leads one to consider the reduced set of subdifferentials which is the set of $x^* \in X^*$ such that there exists $r \in \mathbb{R}$ with $(x^*, r) \in \partial^c f(x)$.

The Greenberg-Pierskalla's subdifferential ∂^*, the Crouzeix's tangential $\partial^T$and several others subdifferentials can be interpreted with the help of this general framework. Let us make more precise the paraconvex case, the case of the Plastria's subdifferential $\partial^<$ and the case of the punctured subdifferential ∂^π.

Example: the paraconvex case ([88], [89], [175], [142], [161], [171], [51]...**).**

Given $r > 0$ and a continuous function $h : X \to I\!R$ (for instance $h(x) := \frac{1}{2}\|x\|^2$), let c_r be the coupling between X and its dual Y given by

$$c_r(x, y) := \langle x, y\rangle - rh(x).$$

Then if f is continuous and h-paraconvex in the sense of section 4, there exists $r > 0$ such that the subdifferential of f with respect to the coupling c_r is nonempty whereas f may have no affine minorant, hence be nowhere subdifferentiable in the Fenchel sense. The corresponding duality theory is linked with what is called the theory of augmented Lagrangians and has a great interest for algorithms.

Example: the conjugacy associated with subaffine functions ([121]**,** [158]**,** [159], [160]). Let us call a function on X *subaffine* or truncated affine if is of the form $y \wedge t := \min(y, t)$ where y is a continuous affine form on X and t is a real number. To this family one can associate the set $Y_T := K_T \circ Y$ of truncated continuous linear forms, where $K_T := \{s_t : t \in I\!R\}$ with $s_t(r) := r \wedge t$ for $r \in I\!R$. Then for $t := \langle y, x\rangle$ one has that the pair (y, t) identified with $s_t \circ y$ belongs to the subdifferential of f at x iff $y \in \partial^< f(x)$ (see [121] Corollary 4.9 and Proposition 4.8 which presents a general characterization of the subdifferential associated with this coupling). Note that this characterization is an immediate consequence of the equivalence

$$\begin{aligned} y &\in \partial^< f(x) \Leftrightarrow f(\cdot) \geq f(x) + \min(\langle y, \cdot - x\rangle, 0) \\ &\Leftrightarrow f(\cdot) \geq f(x) + \min(\langle y, \cdot\rangle, \langle y, x\rangle) - \langle y, x\rangle. \end{aligned}$$

Example: the conjugacy of radiant functions ([3], [151], [152], [197]**,** [198], [199]...). This conjugacy is convenient for functions which take their values in some interval with least element α and have 0 as a minimizer and for problems in which 0 is irrelevant. When $\alpha = -\infty$ it is obtained by taking $K := \{o\}$ where o is the one-variable function given by $o(r) := -\infty$ for $r \leq 1$, $o(r) := 0$ for $r > 1$, so that

$$c^o(x, y) = 0 \text{ if } \langle x, y\rangle > 1, \ -\infty \text{ otherwise}.$$

Then

$$f^o(y) = \sup\{-f(x) : \langle x, y\rangle > 1\},$$

and the associated subdifferential is ∂^o. Similarly, the subdifferential $\partial^\wedge$ is associated with the Atteia-Elqortobi coupling obtained by taking $K := \{\wedge\}$ with $\wedge(r) := -\infty$ for $r < 1$, $\wedge(r) := 0$ for $r \geq 1$ ([3], [159]). The following characterization is given in [213] Théorème 3.4.1 (see also [152]).

Proposition 44 *$f^{oo} = f \Leftrightarrow f$ is l.s.c., quasiconvex and $f(0) = -\infty$.*

These conjugacies can be used for the duality of reverse convex programs and for the maximization of quasiconvex functions ([151], [152], Rubinov and Glover [179], Rubinov and Simsek [180], Thach [197], [198], Tuy [207], Volle [216]...). Similarly, the subdifferentials ∂^∇ and $\partial^\vee$ can be associated with a Fenchel-Moreau duality scheme.

The interest of the subdifferential associated with the coupling c is illustrated by the following result which often gives a characterization of the solution set of the dual optimization problem associated with a perturbation. Recall that a perturbation of the problem

$$(\mathcal{P}) \quad \text{minimize } f(x) \ : \ x \in X$$

defined on an arbitrary set X, is a function $F : W \times X \to \overline{I\!R}$, where W is a normed vector space, such that $F(0,x) = f(x)$ for each $x \in X$. If $c : W \times Y \to \overline{I\!R}$ is a coupling, the dual problem is

$$(\mathcal{D}) \quad \text{maximize } -(p^c(y) - c(0,y)) \ : \ y \in Y,$$

where $p(w) := \inf_{x \in X} F(w,x)$ and p^c is the conjugate of p. Then one has the following result ([159] Prop. 6.1).

Proposition 45 *If the value of the dual problem is finite, then the set S^* of its solutions satisfies*

$$\partial^c p^{cc}(0) = S^* \cap \{y \in Y : c(0,y) \in I\!R\}.$$

Such a result gives an incentive to compute the subdifferential of a performance function and to find conditions ensuring that $p^{cc} = p$.

9 New proposals for transconvex functions

One may consider that the genuine realm of nonsmooth analysis is located in some special favorable classes (see [61], [150], [177], [190]...). Let us devote this last section to what can be considered as a favorable class, the class of transconvex functions, i.e. the class of functions which are deduced from a convex function by a composition. These functions are of two types. The first one is the class of so-called convex composite functions which can be written as $f := h \circ g$ with h convex and g a mapping of class C^1; in the second one g is convex and real-valued and h is a nondecreasing one-variable function. In particular, convex transformable functions belong to the second family.

Since any function f of class C^1 can be put in the form $f = h \circ g$, with $h = I$ the identity mapping of $I\!R$ and $g = f$, one cannot expect any particular property from the specific subdifferentials for the class of convex composite functions. On the other hand, in such a class the usual subdifferentials coincide.

Proposition 46 *Suppose $f := h \circ g$ with h convex, l.s.c., finite at $w := g(x)$ and g a mapping of class C^1 from X into another Banach space W. Suppose*

$$I\!R_+(\mathrm{dom} h - g(x)) - g'(x)(X) = W.$$

Then

$$\partial^- f(x) = \partial^! f(x) = \partial^i f(x) = g'(x)^T(\partial h(w)).$$

Proof. The last two equalities are known (see [144] for instance). It remains to show that any $y \in g'(x)^T(\partial h(w))$ belongs to the Fréchet subdifferential $\partial^- f(x)$.

Let $z \in \partial h(w)$ be such that $y = z \circ g'(x)$. Then for each $u \in X$ we can write $g(x+u) = g(x) + g'(x)(u) + r(u)$ with $\|u\|^{-1} r(u) \to 0$ as $\|u\| \to 0_+$ so that

$$\begin{aligned} f(x+u) - f(x) &\geq \langle z, g(x+u) - g(x) \rangle \\ &\geq \langle z \circ g'(x), u \rangle + \langle z, r(u) \rangle \end{aligned}$$

and $\|u\|^{-1} \langle z, r(u) \rangle \to 0$ as $\|u\| \to 0_+$. Thus $y \in \partial^- f(x)$.□

Now let us turn to functions of the form $f = h \circ g$, with g convex and h nondecreasing and l.s.c.. Is it still possible to propose new adapted concepts for this class?

In view of its interest, let us devote some attention to a recent proposal due to Martinez-Legaz and P.H. Sach [130] (and to a variant of it) as an answer to that question. Their proposal can be viewed as a special case of the scheme described in the preceding section when $x = 0$ (and otherwise one performs a translation, setting $\partial^K f(x) = \partial^K f_x(0)$ where $f_x(u) = f(x+u)$). Here we take for K the set Q of nondecreasing functions q from $\mathbb{R}$ to $\overline{\mathbb{R}}$ such that $q(0) = 0$, q is differentiable at 0 with $q'(0) = 1$. Therefore

$$y \in \partial^Q f(0) \Leftrightarrow \exists q \in Q \quad \forall u \in X \ \ f(u) - f(0) \geq q(\langle y, u \rangle)$$

$$y \in \partial^Q f(x) \Leftrightarrow \exists q \in Q \quad \forall u \in X \ \ f(x+u) - f(x) \geq q(\langle y, u \rangle).$$

The main advantages of this notion are the following : it defines a rather small set $\partial^Q f(x)$ and it is well adapted to the class of quasiconvex functions and to the class of convex-transformable functions. On the other hand $\partial^Q f(x)$ is nonempty only when x is a minimizer of f on some hyperplane containing x, a restrictive requirement. The variant we propose here also suffers from this requirement.

Let us denote by A_X the set of extended real-valued functions α on X such that $\lim_{u \to 0} \alpha(u) = 0$; when X is the set of real numbers we simplify A_X into A.

Then let us introduce the set $\partial^R f(x)$ of $y \in X^*$ such that there exist some $\alpha \in A_X$ for which

$$\forall u \in X \quad f(x+u) - f(x) \geq \langle y, u \rangle + \alpha(u) \langle y, u \rangle$$

This notion is not just a local notion since $0 \in \partial^R f(x)$ iff x is a minimizer of f. Note that $\partial^R f(x)$ is substantially different from $\partial^Q f(x)$ since one has $\partial^Q f(x) \subset \partial^* f(x)$ whereas $\partial^R f(x)$ may contain points outside of $\partial^* f(x)$: for the function f given by $f(r,s) := r(1-s^2)$ one has $(1,0) \in \partial^R f(0,0)$ but $(1,0) \notin \partial^* f(0,0)$.

The fact, proved in [130] Proposition 1.6, that $\partial^Q f$ is quasi-monotone whatever f is, follows from the inclusion $\partial^Q f \subset \partial^* f$. The following lemma shows that in general $\partial^R f$ is not quasi-monotone (take for f a primitive of any non quasi-monotone continuous one variable map which has no zero). However, when f is quasiconvex and u.s.c. on $[f < f(x)]$, the inclusion $\partial^R f \subset \partial^* f$ is a direct consequence of 26 and of the following statement.

Proposition 47 *For any function f and any x in its domain one has*

$$\partial^Q f(x) \subset \partial^R f(x) \subset \partial^- f(x) \subset \partial^! f(x) \subset \partial^i f(x).$$

If f is convex these inclusions are equalities.

Proof. The inclusion $\partial^Q f(x) \subset \partial^R f(x)$ follows from the fact that for any $q \in Q$ the one-variable function ω given by $\omega(0) = 0$,

$$\omega(t) = \frac{1}{t}(q(t) - t) \quad t \neq 0$$

belongs to A and $\alpha := \omega \circ y$ belongs to A_X. The other inclusions are obvious. The last assertion is contained in [130] Proposition 1.2 and in the corollary of the following proposition. □

The difference between $\partial^Q f(x)$ and $\partial^R f(x)$ is enlighten by the case of one variable functions as assertion (b) below is not satisfied with $\partial^Q f(x)$ instead of $\partial^R f(x) : \partial^R f(x)$ is closer to an all-purposes subdifferential than $\partial^R f(x)$.

Lemma 48 *For $f : I\!R \to \overline{I\!R}$ finite at x and for any subdifferential $\partial^?$ one has*
(a) $0 \in \partial^R f(x)$ iff $0 \in \partial^Q f(x)$ iff x is a minimizer of f and then $0 \in \partial^? f(x)$;
(b) if $\partial^? f(x) \subset \partial^! f(x)$ then $\partial^? f(x) \backslash \{0\} \subset \partial^R f(x)$;
(c) if $\partial^? f(x) \supset \partial^- f(x)$ then $\partial^? f(x) \supset \partial^R f(x)$.

Proof. Assertions (a) and (c) have already been observed in the case of a general n.v.s. In order to justify assertion (b) we observe that for any $y \in \partial^? f(x) \backslash \{0\}$, when $y \in \partial^! f(x)$, there exists $\varepsilon(\cdot) \in A$ such that

$$f(x + tv) - f(x) \geq tyv - t\varepsilon(t) \quad \forall t \in I\!R_+$$

for $v = 1, -1$, hence

$$f(x + u) - f(x) \geq yu - yu \mid y^{-1}\varepsilon(u) \mid \quad \forall u \in I\!R. \tag{3}$$

so that $y \in \partial^R f(x)$.□

The following result is close to [130] Prop. 1.4.

Proposition 49 *Let $f = h \circ g$ with g convex, continuous at x and $h : I\!R \to I\!R$ such that $\partial^! h(r) \subset I\!R_+$ for $r := g(x)$. Then*

$$\partial^! h(g(x))\partial g(x) = \partial^- h(g(x))\partial g(x) \subset \partial^- f(x) \subset \partial^! f(x).$$

Moreover, for any $t \in \partial^! h(r) \backslash \{0\}$ one has $t\partial g(x) \subset \partial^R f(x)$.

If h is differentiable at r with $h'(r) \geq 0$ then

$$\partial^! f(x) = \partial^i f(x) = h'(g(x))\, \partial g(x) = \partial^- f(x).$$

If $h'(r) > 0$, these sets are equal to $\partial^R f(x)$. If moreover h is nondecreasing then they are also equal to $\partial^Q f(x)$.

Proof. Let $z \in \partial g(x)$ and let $t \in \partial^! h(r) = \partial^- h(r)$ so that one has for some $\alpha \in A$

$$h(r + s) - h(r) \geq st + s\alpha(s).$$

Since g is continuous, there exist $b > 0, c > 0$ such that for $u \in B(0, b)$ one has

$$\mid g(x + u) - g(x) \mid \leq c\|u\|$$

hence, taking $s := g(x+u) - g(x)$, and using the inequalities $s \geq \langle z, u\rangle$, $t \geq 0$,

$$f(x+u) - f(x) \geq t\langle z, u\rangle - c\|u\|\varepsilon(u)$$

with $\varepsilon(u) :=\mid \alpha(g(x+u) - g(x)) \mid \to 0$ as $u \to 0$. Thus $tz \in \partial^- f(x)$.

When $t > 0$ we have $t + \alpha(s) > 0$ for $\mid s \mid$ small enough, so that, shrinking $B(0, b)$ if necessary, we get

$$\begin{aligned} f(x+u) - f(x) &\geq (g(x+u) - g(x))(t + \alpha(g(x+u) - g(x))) \\ &\geq \langle z, u\rangle (t + \varepsilon(u)) \end{aligned}$$

and $tz \in \partial^R f(x)$. If h is differentiable at r, for each $u \in X$ we have

$$f^! (x, u) = f^i (x, u) = h' (g (x)) g^i (x, u) = h' (g (x)) g' (x, u) .$$

When $h'(r) = h'(g(x)) = 0$, for any $y \in \partial^i f(x)$ we have $y = 0$ and taking an arbitrary $z \in \partial g(x)$ we can write $y = h'(g(x)) z$. When $h'(r) > 0$, for any $y \in \partial^i f(x)$ we get $z := h'(r)^{-1} y \in \partial g(x)$.

The case of a nondecreasing h is given in [130] Proposition 1.4. $\square$

Taking for h the identity mapping of $\mathbb{R}$ we obtain the following consequence.

Corollary 50 *If f is convex, continuous at x then*

$$\partial^Q f(x) = \partial^R f(x) = \partial f(x) .$$

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