Lecture 7
Variational inequalities

Didier Aussel

Univ. de Perpignan

Outline of lecture 7

I- Why V.I.?

II- Existence results

  a- The finite dimension case

  b- The infinite dimention case

  c- The quasimonotone case

III- Other classes of V.I.

IV- Stability/sensitivity

  a- A qualitative result

  b- A quantitative result

I

Why variational inequalities?

First motivation = Optimality conditions

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ and $C \subseteq \operatorname{dom} f$ be a convex subset.

$$(P) \qquad \text{find } \bar{x} \in C \ : \ f(\bar{x}) = \inf_{x \in C} f(x)$$

- Non constrained case: $C = \operatorname{dom} f = X$
- Constrained case

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Perfect case: f convex

$f : X \to \mathbb{R} \cup \{+\infty\}$ a proper convex function

C a nonempty convex subset of X, $\bar{x} \in C$ + C.Q.

Then

$$f(\bar{x}) = \inf_{x \in C} f(x) \quad \Longleftrightarrow \quad \bar{x} \in S(\partial f, C)$$

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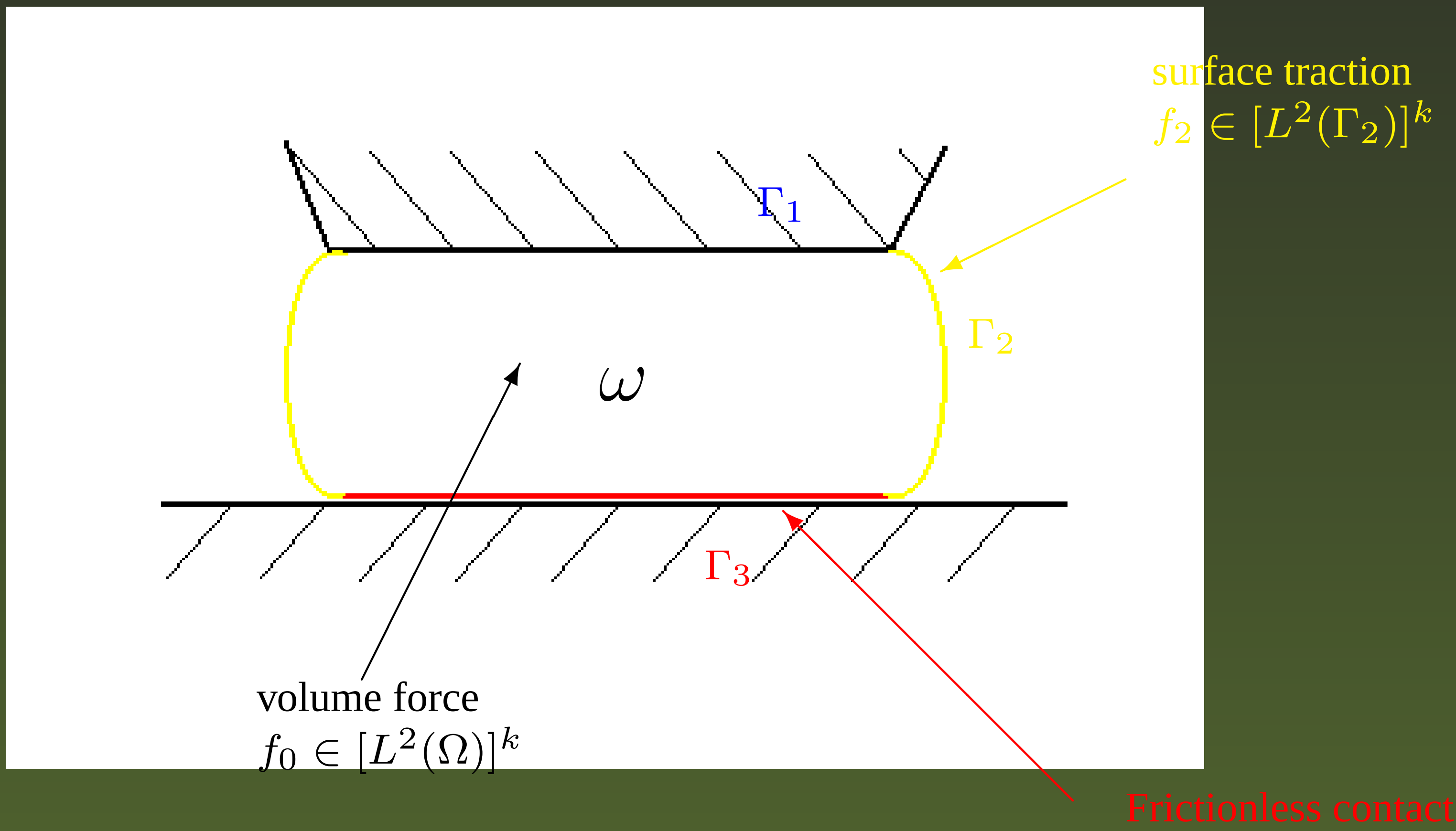
What about quasiconvex case? see Lecture 13

Another motivation

Signorini's frictionless contact problem



Notations

Functional spaces:

$$\mathbb{S}^k = \{\sigma = (\sigma_{ij})_{ij} \in \mathbb{R}^{k\times k} \ : \ \sigma_{ij} = \sigma_{ji}\} = \mathbb{R}_s^{k\times k}$$

$$W = \{v \in H^1(\Omega)^k \ : \ v = 0 \text{ sur } \Gamma_1\}$$

$$Q = \{q = (q_{ij}) \in L^2(\Omega)^{k\times k} \ : \ q_{ij} = q_{ji},\ 1 \le i,j \le k\} = L^2(\Omega)_s^{k\times k}$$

$$W_2 = \{v \in W \ : \ v_\nu \le 0 \text{ a.e. on } \Gamma_3\}$$

Deformation operator: $\varepsilon : H^1(\Omega)^k \to Q$

$$\varepsilon_{ij}(u) = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right], \qquad 1 \le i,j \le k.$$

Scalar product: $\langle p, q\rangle_Q = \int_\Omega p_{ij}(x) q_{ij}(x)\,dx$ et $\langle u, v\rangle_W = \langle \varepsilon(u), \varepsilon(v)\rangle_Q$

Elasticity operator: $\mathcal{F} : \Omega \times \mathbb{S}^k \to \mathbb{S}^k$

Stress function : $\sigma : H^1(\Omega)^k \to Q$ defined by

$$\sigma(u) : \begin{array}{rcl} \Omega & \to & \mathbb{S}^k \\ x & \mapsto & \mathcal{F}(x, \varepsilon(u)(x)). \end{array}$$

Formulations of the problem:

Find a displacement field $u : \Omega \longrightarrow \mathbb{R}$ such that

$$\begin{aligned} -\text{Div}\,\sigma(u) &= f_0 && \text{on} \quad \Omega \\ u &= 0 && \text{on} \quad \Gamma_1 \\ \sigma(u)\nu &= f_2 && \text{on} \quad \Gamma_2 \\ u_\nu \leq 0,\ \sigma(u)_\nu \leq 0,\ \sigma(u)_\nu u_\nu = 0,\ \sigma(u)_\tau = 0 && && \text{on} \quad \Gamma_3 \end{aligned}$$

Variational formulation:

Find $u \in W_2$ such that

$$\langle \sigma(u), \varepsilon(v) - \varepsilon(u) \rangle_Q \geq \langle f, v - u \rangle_W\,, \quad \forall\, v \in W_2$$

where f is an element of W defined by

$$\langle f, v \rangle_W = \int_\Omega f_0 . v\, dx + \int_{\Gamma_2} f_2 . v\, da.$$

II

Existence results

Notations

- X Banach space
- X^* its topological dual (w^*-top.)
- $\langle \cdot, \cdot \rangle$ the duality product

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- X^* its topological dual (w^*-top.)
- $\langle \cdot, \cdot \rangle$ the duality product

Stampacchia variational inequality (strong):

Let $T : X \to 2^{X^*}$ be a map and C be a nonempty subset of X.

Find $\bar{x} \in C$ such that there exists $\bar{x}^* \in T(\bar{x})$ for which

$$\langle \bar{x}^*, y - \bar{x} \rangle \geq 0, \qquad \forall\, y \in C.$$

Notation : $S(T, C)$ set of solutions ($\subset C$).

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Find $\bar{x} \in C$ such that there exists $\bar{x}^* \in T(\bar{x})$ for which

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An equivalent formulation is

$$0 \in T(\bar{x}) + N(C, \bar{x}).$$

II

Existence of solution

a- The finite dimension case

An historical result

Proposition 2 (Stampacchia 1966) *Let C be a nonempty convex compact subset of $\mathbb{R}^n$ and $f : C \to \mathbb{R}^n$ a continuous function. Then $S(f,C)$ is nonempty, that is,*

$$\text{there exists } \bar{x} \in C \text{ such that } \langle f(\bar{x}), y - \bar{x} \rangle \geq 0, \quad \forall\, y \in C.$$

An historical result

Proposition 3 (Stampacchia 1966) *Let C be a nonempty convex compact subset of $\mathbb{R}^n$ and $f : C \to \mathbb{R}^n$ a continuous function. Then $S(f,C)$ is nonempty, that is,*

$$\text{there exists } \bar{x} \in C \text{ such that } \langle f(\bar{x}), y - \bar{x} \rangle \geq 0, \quad \forall\, y \in C.$$

Proof: Let us consider the application

$$\begin{array}{cccl} \psi & C & \longrightarrow & C \\ & x & \mapsto & \psi(x) = P_C \circ (Id - f)(x) = P_C(x - f(x)) \end{array}$$

ψ is continuous on the convex compact set C and therefore, according to the Brouwer fixed point theorem, there exists a point $\bar{x} \in C$ such that

$$\begin{array}{rcl} \bar{x} = \psi(\bar{x}) & \Leftrightarrow & \bar{x} = P_C(\bar{x} - f(\bar{x})) \\ & \Leftrightarrow & \langle \bar{x} - f(\bar{x}) - \bar{x}, y - \bar{x} \rangle \leq 0, \ \forall\, y \in C \\ & \Leftrightarrow & \bar{x} \in S(f,C). \ \blacksquare \end{array}$$

Without compacity

The compacity hypothesis can be replaced by a "coercivity condition":

Proposition 4 *Let C be a nonempty closed convex subset of $\mathbb{R}^n$ and $f : C \to \mathbb{R}^n$ a continuous function. If the following condition holds*

$$\exists\, r > 0 \text{ such that } S(f, C \cap \overline{B}(0,r)) \cap B(0,r) = \emptyset,$$

then $S(f, C)$ is nonempty.

Multivalued form

Theorem 6 *Let C be a nonempty convex compact subset of $\mathbb{R}^n$ and $F : C \to 2^{\mathbb{R}^n}$ a upper semicontinuous map with convex compact values. Then $S(F, C)$ is nonempty, that is,*

$$\text{there exist } \bar{x} \in C \text{ and } \bar{x}^* \in F(\bar{x})$$

$$\text{such that } \langle \bar{x}^*, y - \bar{x} \rangle \geq 0, \quad \forall\, y \in C.$$

Multivalued form

Theorem 7 *Let C be a nonempty convex compact subset of $\mathbb{R}^n$ and $F : C \to 2^{\mathbb{R}^n}$ a upper semicontinuous map with convex compact values. Then $S(F, C)$ is nonempty, that is,*

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$$\textit{such that } \langle \bar{x}^*, y - \bar{x} \rangle \geq 0, \quad \forall\, y \in C.$$

F is upper semicontinuous at x if

$\forall\, V$ neighb. of $F(x)$, $\exists\, U$ neighb. of x such that $F(U) \subset V$.

Multivalued form

Theorem 8 *Let C be a nonempty convex compact subset of $\mathbb{R}^n$ and $F : C \to 2^{\mathbb{R}^n}$ a upper semicontinuous map with convex compact values. Then $S(F, C)$ is nonempty, that is,*

$$\text{there exist } \bar{x} \in C \text{ and } \bar{x}^* \in F(\bar{x})$$

$$\text{such that } \langle \bar{x}^*, y - \bar{x} \rangle \geq 0, \quad \forall\, y \in C.$$

The proof is based on the Kakutani Fixed point theorem:

> *If C is a nonempty convex compact subset of $\mathbb{R}^n$ and $T : C \to 2^C$ is a cusco map, then T admits a fixed point.*

Multivalued form: proof

Proof. Since F is cusco, its range $F(C)$ is compact and the map $T = (Id - F) \circ P_C$ is also cusco since P_C is continuous.
Now one can observe that $T(C - F(C)) = C - F(C)$. Therefore T has a fixed point (say x_0) on $C - F(C)$, that is

$$
\begin{aligned}
x_0 \in T(x_0) \quad &\Leftrightarrow \quad x_0 \in (Id - F) \circ P_C(x_0) \\
&\Leftrightarrow \quad \begin{cases} \bar{x} = P_C(x_0) \\ \bar{x}^* = \bar{x} - x_0 \in F(\bar{x}) \end{cases} \\
&\Leftrightarrow \quad \begin{cases} \langle x_0 - \bar{x}, y - \bar{x} \rangle \leq 0, \quad \forall\, y \in C \\ \bar{x}^* = \bar{x} - x_0 \in F(\bar{x}) \end{cases}
\end{aligned}
$$

and thus $\bar{x} \in S(F, C)$. ■

Equivalence

Actually the existence theorem 6 is equivalent to the Kakutani Fixed point theorem. Indeed, let C be a nonempty convex compact subset of $\mathbb{R}^n$ and $T : C \longrightarrow 2^C$ be a cusco map. Observing that the set-valued map $F = Id - T$ is also cusco, Theorem 6 implies the solvability of the set-valued Stampacchia variational inequality defined by F and C, that is there exists $\bar{x} \in C$ such that

$$\bar{x} \in S(Id - T, C) \quad \Leftrightarrow \quad \begin{cases} \exists\, u^* \in (Id - T)(\bar{x}) \text{ such that} \\ \langle u^*, y - \bar{x} \rangle \geq 0, \ \forall\, y \in C \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} \exists\, \bar{x}^* \in T(\bar{x}) \text{ such that} \\ \langle \bar{x} - \bar{x}^*, y - \bar{x} \rangle \geq 0, \ \forall\, y \in C \end{cases}$$

Taking $y = \bar{x}^*$, one has

$$\langle \bar{x} - \bar{x}^*, \bar{x}^* - \bar{x} \rangle \geq 0$$

showing that $\bar{x}$ is a fixed point of T. ■

II

Existence of solution

b- The infinite dimension case

Let H be an Hilbert space, f an element of $H^* = H$ and $a : H \times H \longrightarrow H$ a bilinear application.

Theorem 9 (Stampacchia 64) *Let C be a nonempty closed convex subset of H. If a is continuous and satisfies the following property*

$$\exists\, \alpha > 0 \textit{ such that } a(v,v) \geq \alpha \|v\|^2, \;\; \forall\, v \in H,$$

then

$$\textit{there exists } \bar{x} \in C \textit{ such that } a(\bar{x}, y - \bar{x}) \geq \langle f, y - \bar{x} \rangle, \quad \forall\, y \in C.$$

Theorem 10 (Lax-Milgram) *If a is continuous and satisfies the following property*

$$\exists\, \alpha > 0 \text{ such that } a(v,v) \geq \alpha \|v\|^2, \quad \forall\, v \in H,$$

then there exists a unique $\bar{x} \in H$ such that

$$a(\bar{x}, u) = \langle f, u \rangle, \quad \forall\, u \in H.$$

Proof. From Theorem 9 there exists $u \in H$ such that

$$a(u, v-u) \geq \langle f, v-u \rangle, \quad \forall\, v \in H$$

or in other words $\varphi_u(v) \geq \varphi_u(u), \quad \forall\, v \in H$ where

$$\varphi_u(v) = a(u,v) - \langle f, v \rangle.$$

But φ_u is linear and therefore $\varphi_u \equiv 0$. ■

Theorem 12 (Shih & Tan 1988) *Let X be a reflexive Banach space, C be a nonempty convex and weakly compact subset of X. If $F : C \to 2^{X^*}$ is a monotone map with convex weakly compact values and continuous on finite dimension subsets, then $S(F, C)$ is nonempty.*

Theorem 15 (Shih & Tan 1988) *Let X be a reflexive Banach space, C be a nonempty convex and weakly compact subset of X. If $F : C \to 2^{X^*}$ is a monotone map with convex weakly compact values and continuous on finite dimension subsets, then $S(F, C)$ is nonempty.*

Theorem 16 (Yao 1994) *Let X be a reflexive Banach space, C be a nonempty convex and weakly compact subset of X. If $F : C \to 2^{X^*}$ is a pseudomonotone map with convex weakly compact values and continuous on finite dimension subsets, then $S(F, C)$ is nonempty.*

Monotonicities

- T is monotone iff $\forall\, x, y \in X, \forall\, x^* \in T(x)$ and $\forall\, y^* \in T(x)$

$$\langle y^* - x^*, y - x \rangle \geq 0.$$

- T is pseudomonotone iff

$$\exists\, x^* \in T(x) \;:\; \langle x^*, y - x \rangle \geq 0 \Rightarrow \langle y^*, y - x \rangle \geq 0, \;\forall\, y^* \in T(y).$$

monotone

$\Downarrow$

pseudomonotone

II

Existence of solution

C- The quasimonotone case

- T is monotone iff $\quad \forall\, x, y \in X, \forall\, x^* \in T(x)$ and $\forall\, y^* \in T(x)$

$$\langle y^* - x^*, y - x\rangle \geq 0.$$

- T is pseudomonotone iff

$$\exists\, x^* \in T(x)\ :\ \langle x^*, y - x\rangle \geq 0 \Rightarrow \langle y^*, y - x\rangle \geq 0,\ \forall\, y^* \in T(y).$$

- T is quasimonotone iff

$$\exists\, x^* \in T(x)\ :\ \langle x^*, y - x\rangle > 0 \Rightarrow \langle y^*, y - x\rangle \geq 0,\ \forall\, y^* \in T(y).$$

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$\Downarrow$

pseudomonotone

$\Downarrow$

quasimonotone

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- T is quasimonotone iff

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Some previous attempts:

- Dinh The Luc (2001) with quasimonotone, densely pseudomonotone maps
- N. Hadjisavvas and S. Schaible (1996) with quasimonotone + existence of inner points

Existence: simplified form

Theorem 17

C nonempty convex compact subset of X.

$T : C \to 2^{X^*}$ quasimonotone

+ upper hemi-continuous

+ $T(x) \neq \emptyset$ convex w*-compact.

Then $S(T, C) \neq \emptyset$.

T is upper hemicontinuous on X if the restriction of T to any line segment is usc with respect to the w^*-topology

Knaster-Kuratowski-Mazurkiewicz (1929) :

C nonempty subset of X.

$\varphi : C \times C \longrightarrow \mathbb{R}$ is said to be a *KKM-application* if

$$\forall\, x_1, \ldots, x_n \in C, \ \ \forall\, x \in \text{co}\{x_1, \ldots, x_n\}$$

$$\exists\, i \in 1, \ldots, n \quad \text{such that } \varphi(x_i, x) \geq 0.$$

Theorem 18

C nonempty closed convex subset of X.

$\varphi : C \times C \longrightarrow \mathbb{R}$ KKM-application

+ $\varphi(x, .)$ usc quasiconcave, $\forall\, x$

+ $\exists\, \tilde{x} \in C$ with $\{y \in C \ : \ \varphi(\tilde{x}, y) \geq 0\}$ compact.

Then $\exists\, \bar{y} \in C$ *such that* $\varphi(x, \bar{y}) \geq 0, \forall\, x \in C.$

A very simple proof:

Let us define $\varphi : \quad C \times C \to \quad \mathbb{R}$

$$(x,y) \to \quad \inf_{x^* \in T(x)} \langle x^*, x-y \rangle$$

The function φ is continuous concave relatively to y and $\{y \in C \ : \ \varphi(x,y) \geq 0\}$ is compact, $\forall\, x \in C$.

<u>CAS 1 :</u> φ is KKM $\overset{Th.KKM}{\Longrightarrow} \exists\, \bar{y} \in C$ such that $\varphi(x,\bar{y}) \geq 0, \ \ \forall\, x \in C$.

<u>CAS 2 :</u> $\exists\, x_1,\ldots,x_n \in C, \exists\, \bar{y} \in co\{x_1,\ldots,x_n\}$ such that

$$\varphi(x_i,\bar{y}) < 0, \quad \forall\, i = 1,\ldots,n$$

that is $\quad \forall\, i = 1,\ldots,n, \ \exists\, x_i^* \in T(x_i) \ : \ \langle x_i^*, \bar{y} - x_i \rangle > 0$

and, for some $\rho > 0$ and all $z \in B(\bar{y},\rho) \cap C$

$$\forall\, i, \ \ \exists\, x_i^* \in T(x_i) \ : \ \langle x_i^*, z - x_i \rangle > 0.$$

By quasimonotonicity,

$$\forall\, z \in B(\bar{y},\rho) \cap C, \quad \forall\, z^* \in T(z), \quad \forall\, i = 1,\ldots,n, \langle z^*, z - x_i \rangle \geq 0.$$

In both cases,

$$\forall\, z \in B(\bar{y},\rho) \cap C, \;\; \forall\, z^* \in T(z), \quad \langle z^*, z - \bar{y} \rangle \geq 0. \qquad (1)$$

<u>Let us prove that $\bar{y} \in S(T,C)$:</u>

Let $x \in C$ $(x \neq \bar{y})$, $t_0 > 0$ be such that $z_t = \bar{y} + t(x - \bar{y}) \in B(\bar{y},\rho) \cap C$ for all $t \in]0, t_0[$. From (1), one have,

$$\langle z_t^*, x - \bar{y} \rangle \geq 0, \quad \forall\, t \in]0, t_0[, \;\; \forall\, z_t^* \in T(z_t). \qquad (2)$$

Let us define the open subset

$$W = \{y^* \in X^* \; : \; \langle y^*, x - \bar{y} \rangle < 0\}.$$

If $T(\bar{y}) \subset W$, by upper hemicontinuity, $T(z_t) \subset W$, for any t sufficiently closed to 0, which is a contradiction with (2).

Finally $\exists\, \bar{y}^* \in T(\bar{y}) \;:\; \langle \bar{y}^*, x - \bar{y} \rangle \geq 0$, that is

$$\inf_{x \in C} \sup_{y^* \in T(\bar{y})} \langle y^*, x - \bar{y} \rangle \geq 0$$

thus $\bar{y} \in S(T, C)$ by a classical minimax theorem (Sion). $\square$

Upper sign-continuity

- $T : X \to 2^{X^*}$ is said to be *upper sign-continuous* on C iff for any $x, y \in C$, one have :

$$\forall\, t \in]0,1[, \qquad \inf_{x^* \in T(x_t)} \langle x^*, y - x \rangle \geq 0$$

$$\Longrightarrow \sup_{x^* \in T(x)} \langle x^*, y - x \rangle \geq 0$$

where $x_t = (1-t)x + ty$.

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upper semi-continuous

$\Downarrow$

upper hemicontinuous

$\Downarrow$

upper sign-continuous

locally upper sign continuity

Definition 18 *Let $T : C \to 2^{X^*}$ be a set-valued map.*

T *est called* locally upper sign-continuous *on* C *if, for any* $x \in C$ *there exist a neigh.* V_x *of* x *and a upper sign-continuous set-valued map* $\Phi_x : V_x \to 2^{X^*}$ *with nonempty convex* w^**-compact values such that* $\Phi_x(y) \subseteq T(y) \setminus \{0\}, \forall\, y \in V_x$

Existence: complete form

C nonempty convex compact subset of X.

$T : C \to 2^{X^*}$ quasimonotone

+ $T(x) \neq \emptyset$ convex w*-compact

+ upper hemicontinuous

Then $S(T, C) \neq \emptyset$.

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Existence: complete form

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$T : C \to 2^{X^*}$ quasimonotone

+ locally upper sign continuous on C

+ coercivity condition:

$\exists\, \rho > 0,\ \forall\, x \in C \setminus \overline{B}(0, \rho),\ \exists\, y \in C$ with $\|y\| < \|x\|$

such that $\forall\, x^* \in T(x), \langle x^*, x - y\rangle \geq 0$

and there exists $\rho' > \rho$ such that $C \cap \overline{B}(0, \rho')$ is weakly

compact ($\neq \emptyset$).

Then $\quad S(T, C) \neq \emptyset$.

III

Other classes of V.I.

Other V. I.

Let $C \subseteq X$ and $T : C \to 2^{X^*}$ be a map.

We denote by

- $S_w(T, C)$ set of weak solutions of Stampacchia V.I.

  Find $\bar{x} \in C$ such that

$$\forall\, y \in C,\ \exists\, \bar{x}^* \in T(\bar{x}) : \ \ \langle \bar{x}^*, y - \bar{x} \rangle \geq 0$$

Other V. I.

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- $M(T, C)$ set of solutions of the Minty V.I.

  Find $\bar{x} \in C$ such that

$$\forall\, y \in C \text{ and } \ \forall\, y^* \in T(y) \ : \ \langle y^*, y - \bar{x} \rangle \geq 0.$$

Other V. I.

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- $M(T, C)$ set of solutions of the Minty V.I.

  Find $\bar{x} \in C$ such that

$$\forall\, y \in C \text{ and } \ \forall\, y^* \in T(y) \ : \ \langle y^*, y - \bar{x} \rangle \geq 0.$$

- $LM(T, C)$ set of local solutions of Minty V.I., i.e.

$\bar{x} \in LM(T, C)$ if it exists a neighb. V of $\bar{x}$ such that $\bar{x} \in M(T, V \cap C)$.

Even more general

X, Y l.c.t.v.s., C subset of X

$T : C \longrightarrow Y$ set-valued map

Coupling function : $\quad \Phi : Y \times C \longrightarrow \mathbb{R}$

- Variational inequality:

  Find $\bar{x} \in C$ and $\bar{x}^* \in T(\bar{x})$ such that

$$\Phi(\bar{x}^*, y) - \Phi(\bar{x}^*, \bar{x}) \geq 0, \quad \forall\, y \in C.$$

Even more general

X, Y l.c.t.v.s., C subset of X

$T : C \longrightarrow Y$ set-valued map

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  Find $\bar{x} \in C$ and $\bar{x}^* \in T(\bar{x})$ such that

$$\Phi(\bar{x}^*, y) - \Phi(\bar{x}^*, \bar{x}) \geq 0, \quad \forall\, y \in C.$$

Example : Mixed var. ineq.

Find $\bar{x} \in C$ and $\bar{x}^* \in T(\bar{x})$ such that

$$\langle \bar{x}^*, y - \bar{x} \rangle + h(y) - h(\bar{x}) \geq 0, \quad \forall\, y \in C.$$

Also called *hemi-variational inequality*

Even more general

X, Y l.c.t.v.s., C subset of X

$T : C \longrightarrow Y$ set-valued map

Coupling function : $\Phi : Y \times C \longrightarrow \mathbb{R}$

- Variational inequality:

Find $\bar{x} \in C$ and $\bar{x}^* \in T(\bar{x})$ such that

$$\Phi(\bar{x}^*, y) - \Phi(\bar{x}^*, \bar{x}) \geq 0, \quad \forall\, y \in C.$$

Example : General var. ineq.

Find $x_0 \in X$ such that

$$\langle f(x_0), g(x) - g(x_0) \rangle \geq 0, \quad \forall\, x \in X \text{ avec } g(x) \in C.$$

Even more general

X, Y l.c.t.v.s., C subset of X

$T : C \longrightarrow Y$ set-valued map

Coupling function : $\quad \Phi : Y \times C \longrightarrow \mathbb{R}$

- Variational inequality:

  Find $\bar{x} \in C$ and $\bar{x}^* \in T(\bar{x})$ such that

$$\Phi(\bar{x}^*, y) - \Phi(\bar{x}^*, \bar{x}) \geq 0, \quad \forall\, y \in C.$$

Example : Equilibrium problem

Find $x_0 \in C$ such that

$$f(x_0, x) + h(x) - h(x_0) \geq 0, \quad \forall\, x \in C.$$

Even more general

X, Y l.c.t.v.s., C subset of X

$T : C \to Y$ set-valued map

Coupling function : $\Phi : Y \times C \to \mathbb{R}$

- Variational inequality:

  Find $\bar{x} \in C$ and $\bar{x}^* \in T(\bar{x})$ such that

$$\Phi(\bar{x}^*, y) - \Phi(\bar{x}^*, \bar{x}) \geq 0, \quad \forall\, y \in C.$$

- Quasimonotonicity –> Φ-quasimonotonicity
- Upper-sign continuity –> Φ-upper sign-continuity

Even more general

X, Y l.c.t.v.s., C subset of X

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$$\Phi(\bar{x}^*, y) - \Phi(\bar{x}^*, \bar{x}) \geq 0, \quad \forall\, y \in C.$$

- Quasimonotonicity –> Φ-quasimonotonicity
- Upper-sign continuity –> Φ-upper sign-continuity

Hyp.: $\Phi(x^*, \cdot)$ convexe continue and $\Phi(\cdot, x)$ concave.

See Aussel-Luc, Bull. Austral. Math. Soc. (2005)

Links ?

Proposition 19 *C nonempty convex subset of X and $T : C \to 2^{X^*}$ a map.*

- *i) If T is pseudomonotone, then $LM\,(T, C) = M(T, C)$.*
- *ii) If T is locally upper sign-continuous at x then $LM(T, C) \subseteq S_w(T, C)$.*
- *iii) If, additionally to $ii)$, the maps S_x are convex valued, then $LM(T, C) \subseteq S_w(T, C) = S(T, C)$.*

Proper Quasimonotonicity:

Let $C \subseteq X$ and $T : C \to 2^{X^*}$.

- T is *properly quasimonotone* if, for any $x_1, \ldots, x_n \in \operatorname{dom} T$, and any $x \in co\{x_1, x_2, \ldots, x_n\}$, there exists $i \in \{1, 2, \ldots, n\}$ such that

$$\forall x^* \in T(x_i) : \langle x^*, x_i - x \rangle \geq 0.$$

monotone

$\Downarrow$

pseudomonotone

$\Downarrow$

properly quasimonotone

$\Downarrow$

quasimonotone

Existence of Minty V.I.

Proposition 20 (R. John (1999))
Let $T : X \to 2^{X^*}$ be a set-valued map. Then the following are equivalent:

i) T *is properly quasimonotone*

ii) $\forall C$ *convex compact,* $M(T, C) \neq \emptyset$.

The proof is based on the Ky Fan theorem:

> *Let C be a closed set of X and $D \subset C$. If $\{C_x : x \in D\}$ is a KKM family of closed convex subsets of C and one of them is compact, then $\cap\{C_x : x \in D\}$ is nonempty.*

Existence of Minty V.I.

Proof of $i) \Rightarrow ii)$.
Let us define the set-valued map $G : C \longrightarrow C$ by

$$G(y) = \{x \in C \ : \ \forall\, y^* \in T(y), \langle y^*, y - x\rangle \geq 0\}.$$

By hypothesis, $G(y)$ is nonempty, convex and compact. Moreover the map G is KKM in the sense that for any $x_1, \ldots, x_n \in C$, one has $conv\{x_1, \ldots, x_n\} \subset \cup_{i=1,n} G(x_i)$ which is immediate from the definition of proper quasimonotonicity. By applying Ky Fan's theorem, we have

$$\cap_{y \in C} G(y) \neq \emptyset.$$

But any element of this intersection (say $\bar{x}$) satisfies

$$\forall\, y \in C, \ \ \forall\, y^* \in T(y), \quad \langle y^*, y - \bar{x}\rangle \geq 0$$

and is therefore a solution of the Minty variational inequality.

For the converse, see R. John, Proc. of the GCM7 (1999). ■

Existence of local solutions of Minty V.I.

Proposition 21 *Let C be a nonempty convex subset of X and $T : C \to 2^{X^*}$ be quasimonotone. Then one of the following assertions holds:*

i) T *is properly quasimonotone*

ii) $LM(T, C) \neq \emptyset$.

If, C is weakly compact, then $LM\,(T, C) \neq \emptyset$ in both cases.

Existence of local solutions of Minty V.I.

Proof. Suppose that $i)$ does not hold. Thus there exists $x_1, \ldots, x_n \in C$, $x_i^* \in T(x_i)$ and $x \in conv\{x_1, \ldots, x_n\}$ such that

$$\langle x_i^*, x - x_i \rangle > 0, \quad \forall\, i.$$

By continuity of the x_i^*'s, there exists a neighb. U of x for which

$$\langle x_i^*, x - x_i \rangle > 0, \quad \forall\, y \in C \cap U$$

and thus, according to the quasimonotonicity of T, for any $y^* \in T(y)$,

$$\langle y^*, y - x_i \rangle \geq 0.$$

Now since $x \in conv\{x_1, \ldots, x_n\}$, for any $y \in C \cap U$ and any $y^* \in T(y)$, one has $\langle y^*, y - x \rangle \geq 0$ and $x \in LM(T, C)$. ■

IV - Stability of Variational inequalities

Setting

- X Banach space
- X^* its topological dual (w^*-top.)
- $\langle \cdot, \cdot \rangle$ the duality pairing

Setting

- X Banach space
- X^* its topological dual (w^*-top.)
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Stampacchia variational inequality:

Let $T : X \to 2^{X^*}$ is a set-valued map
C is a nonempty set of X.

Find $\bar{x} \in C$ *such that there exists* $\bar{x}^* \in T(\bar{x})$ *with*

$$\langle \bar{x}^*, y - \bar{x} \rangle \geq 0, \qquad \forall\, y \in C.$$

Setting

- X Banach space
- X^* its topological dual (w^*-top.)
- $\langle \cdot, \cdot \rangle$ the duality pairing

Stampacchia variational inequality:

Let U, Λ normed spaces
$T : X \times \Lambda \to 2^{X^*}$ is a set-valued map
$C : U \to 2^X$ is a set-valued map.

$(P_{(\lambda,\mu)})$ *Find* $\bar{x} \in C(\mu)$ *and* $\bar{x}^* \in T(\bar{x}, \lambda)$ *such that*

$$\langle \bar{x}^*, y - \bar{x} \rangle \geq 0, \quad \forall\, y \in C(\mu).$$

Outline

Aim : In the quasimonotone setting, "measure" the influence of perturbations on the solution set.

Outline

Aim : In the quasimonotone setting, "measure" the influence of perturbations on the solution set.

a- Qualitative approach (usc, lsc)

b- Quantitative approach (Holder type estimation)

Outline

Aim : In the quasimonotone setting, "measure" the influence of perturbations on the solution set.

a- Qualitative approach (usc, lsc)

with T monotone or pseudomonotone: Bianchi-Pini (03), Chu-Lin (97), Domokos (99), Gwinner (95), Lignola-Morgan (99), ...

b- Quantitative approach (Holder type estimation)

with T strongly monotone or strongly pseudomonotone: Attouch-Wets (93), Bianchi-Pini (03), Yen (95), ...

Solution maps (set-valued)

- Classical Solutions

$$S(\lambda,\mu) = \{x \in C(\mu)\ :\ \exists\, x^* \in T(x,\lambda) \text{ with } \langle x^*, y - x\rangle \geq 0, \quad \forall\, y \in C(\mu)\}\,.$$

Solution maps (set-valued)

- Classical Solutions

$$S(\lambda, \mu) = \{x \in C(\mu) \; : \; \exists\, x^* \in T(x, \lambda) \text{ with } \langle x^*, y - x \rangle \geq 0, \quad \forall\, y \in C(\mu)\}\,.$$

Weakening monotonicity $\Rightarrow$ need of other solution maps

Solution maps (set-valued)

- Classical Solutions

$$S(\lambda, \mu) = \{x \in C(\mu) \ : \ \exists\, x^* \in T(x, \lambda) \text{ with } \langle x^*, y - x \rangle \geq 0, \quad \forall\, y \in C(\mu)\}\,.$$

- Strict Solutions

$x \in S^{>}(\lambda, \mu) \Longleftrightarrow x \in C(\mu)$ and

$$\exists\, x^* \in T(x, \lambda) \text{ with } \langle x^*, y - x \rangle > 0, \quad \forall\, y \in C(\mu) \setminus \{x\}$$

- Star-solutions

$x \in S^{*}(\lambda, \mu) \Longleftrightarrow x \in C(\mu)$ and

$$\exists\, x^* \in T(x, \lambda) \setminus \{0\} \text{ with } \langle x^*, y - x \rangle \geq 0, \quad \forall\, y \in C(\mu)$$

Solution maps (set-valued)

- Classical Solutions

$$S(\lambda,\mu) = \{x \in C(\mu) \ : \ \exists\, x^* \in T(x,\lambda) \text{ with } \langle x^*, y - x\rangle \geq 0, \quad \forall\, y \in C(\mu)\}\,.$$

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- Star-solutions

$x \in S^{*}(\lambda,\mu) \Longleftrightarrow x \in C(\mu)$ and

$$\exists\, x^* \in T(x,\lambda) \setminus \{0\} \text{ with } \langle x^*, y - x\rangle \geq 0, \quad \forall\, y \in C(\mu)$$

$$\boxed{S^{>}(\lambda,\mu) \subseteq S^{*}(\lambda,\mu) \subseteq S(\lambda,\mu)}$$

a

Qualitative approach

Set Convergence

Definition 22

A sequence $(C_n)_n$ of subsets of X Mosco-converges to a subset C if

$$w - \operatorname{Lim}\sup_n C_n \subset C \quad \textit{and} \quad C \subset \operatorname{Lim}\inf_n C_n.$$

Painvelé-Kuratowski limits :

$$\operatorname{Lim}\inf_n C_n = \{x \in X \ : \ \lim_{n \to \infty} d(x, C_n) = 0\}$$

$$\operatorname{Lim}\sup_n C_n = \{x \in X \ : \ \liminf_{n \to \infty} d(x, C_n) = 0\}.$$

Mosco-convergence: a good example

Let us consider, for any $\mu \in \mathbb{R}^q$, the subset

$$C(\mu) = \{x \in X \;:\; g_i(x) \le \mu^i, \;\; \forall\, i = 1, \ldots, q\}.$$

where $g_i : X \to \mathbb{R}$, for $i = 1, \ldots, q$.

In general, $\mu_n \to \mu \;\; \not\Longrightarrow \;\; C(\mu_n) \to C(\mu)$.

Mosco-convergence: a good example

Let us consider, for any $\mu \in \mathbb{R}^q$, the subset

$$C(\mu) = \{x \in X \ : \ g_i(x) \leq \mu^i, \ \forall\, i = 1, \ldots, q\}.$$

where $g_i : X \to \mathbb{R}$, for $i = 1, \ldots, q$.

Proposition 22 *If $\mu_n \to \mu$ and*

$i)$ g_i *are continuous and semistrictly quasiconvexe on* X

$ii)$ $\mathrm{int}(C(\mu)) \neq \emptyset$ *and* $\mathrm{int}(C(\mu_n)) \neq \emptyset, \ \forall\, n$

Then $C(\mu_n)$ Mosco-converges to $C(\mu)$.

Closed graph (perturbation on C)

Theorem 23

Suppose $S^*(\mu) \neq \emptyset$ for all $\mu \in U$, and

- *i) for any $\mu \in U$, $C(\mu)$ is convex with nonempty interior;*
- *ii) $T(\cdot)$ is quasimonotone and locally upper sign-continuous;*
- *iii) for any $\mu_n \to \mu$, $C(\mu_n)$ Mosco-converges to $C(\mu)$.*

Then the graph of S^* is (norm-w^*) closed.

Closed graph (perturbation on C and T single-valued)

Theorem 24

Suppose $S^*(\lambda, \mu) \neq \emptyset$ *for all* $(\lambda, \mu) \in \Lambda \times U$*, and*

i) *for any* $\mu \in U$*,* $C(\mu)$ *is convex with nonempty interior;*

ii) *for any* $(\lambda, \mu) \in \Lambda \times U$*,* $T(\cdot, \lambda)$ *is quasimonotone and locally upper sign-continuous on* $C(\mu)$*;*

iii) *for any* $\mu_n \to \mu$*,* $C(\mu_n)$ *Mosco-converges to* $C(\mu)$*.*

iv) *application* $(y, \lambda) \to T(y, \lambda) \setminus \{0\}$ *is lower semicontinuous;*

Then the graph of S^* *is (norm-*w^**) closed.*

Closed graph (perturbation on C and T)

Theorem 25

Suppose $S^*(\lambda,\mu) \neq \emptyset$ *for all* $(\lambda,\mu) \in \Lambda \times U$, *and*

i) *for any* $\mu \in U$, $C(\mu)$ *is convex with nonempty interior;*

ii) *for any* $(\lambda,\mu) \in \Lambda \times U$, $T(\cdot,\lambda)$ *is quasimonotone and locally upper sign-continuous on* $C(\mu)$*;*

iii) *for any* $\mu_n \to \mu$, $C(\mu_n)$ *Mosco-converges to* $C(\mu)$.

iv) *for any* $y_n \to y$, *any* $z_n \to z$ *and any* $\lambda_n \to \lambda$,

$$\sup_{y^* \in T(y,\lambda)\setminus\{0\}} \langle y^*, z - y\rangle \leq \liminf_n \sup_{y_n^* \in T(y_n,\lambda_n)\setminus\{0\}} \langle y_n^*, z_n - y_n\rangle;$$

Then the graph of S^* *is (norm-*w^**) closed.*

Upper semicontinuity

Corollary 26 *If, moreover, the subset* $C(U) = \cup_{\mu \in U} C(\mu)$ *is compact, then the set-valed map* S^* *is upper semicontinuous on* $\Lambda \times U$.

that is,

$\forall V$ neigh. of $S^*(x)$, $\exists U$ neigh. of x such that $S^*(U) \subset V$.

II

Quantitative approach

Strong quasimonotonicity

A set-valued map $T : X \longrightarrow 2^{X^*}$ is said t be:

- (α)-*strongly quasimonotone* if, for any x, $y \in X$

$$\exists\, x^* \in T(x) \;:\; \langle x^*, y - x\rangle > 0 \Rightarrow \langle y^*, y - x\rangle \geq \alpha\|y - x\|^2, \quad \forall\, y^* \in T(y).$$

A set-valued map $T : X \times \Lambda \longrightarrow 2^{X^*}$ is said to be:

- uniformly (α)-*strongly quasimonotone* on $\mathcal{V} \subset \Lambda$ if,

  $T(\cdot, \lambda)$ is (α)-*strongly quasimonotone* for any $\lambda \in \mathcal{V}$.

Perturbation on T

Theorem 27 *Let C be a convex bounded subset of X with nonempty interior, $\mathcal{V}_0$ a neigh. of $\lambda_0 \in \Lambda$ and $T : X \times \Lambda \to 2^{X^*}$ uniformly α-strongly quasimonotone on $\mathcal{V}_0$ with nonempty bounded values.*
Suppose

i) $\lambda \mapsto T(.,\lambda)$ *is uniformly (in x) Hölder continuous i.e.* $\exists \beta > 0, \eta > 0$:

$$\text{haus}(T(x,\lambda), T(x,\lambda')) \leq \beta \|\lambda - \lambda'\|^{\eta}, \forall\, \lambda, \lambda' \in \mathcal{V}_0,\ \forall\, x \in C;$$

ii) *For any $\lambda \in \mathcal{V}_0$, any $x, y \in C$, and any sequence $x_n \to x$,*

$$\sup_{x^* \in T(x,\lambda)} \langle x^*, y - x\rangle \leq \liminf_n \sup_{x_n^* \in T(x_n,\lambda)} \langle x_n^*, y - x_n\rangle.$$

iii) $S^*(\lambda)$ *is nonempty for all $\lambda \in \mathcal{V}_0$.*

Then

$$\text{haus}\,(S^*(\lambda), S^*(\lambda')) \leq \frac{\beta}{\alpha}\|\lambda - \lambda'\|^{\gamma}, \qquad \forall \lambda, \lambda' \in \mathcal{V}_0.$$

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