Lecture 16 (second part)

Generalized Nash Equilibrium Problem

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Outline of lecture 16 (second part)

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III- Reformulations of GNEP

- Classical ones
- An extented reformulation

IV- Special important cases

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I

The model

Noncooperative multi-leader-follower games

The generalized Nash equilibrium problem (GNEP) is a noncooperative game is which each player's admissible strategy set depends on the other players' strategies.

More precisely, assume that there are N players and each player ν controls variables $x^\nu \in \mathbb{R}^{n_\nu}$. In fact x^ν is a strategy of the player ν.

Let us denote by x the following vector

$$x = \begin{pmatrix} (x^1) \\ \vdots \\ (x^N) \end{pmatrix}$$

and let us set $n = n_1 + n_2 + \cdots + n_N$. Thus $x \in \mathbb{R}^n$.

Denote by $x^{-\nu}$ the vector formed of all players decision variables except the one of the player ν. So we can also write

$$x = (x^{\nu}, x^{-\nu}).$$

The strategy of the player ν belongs to a strategy set

$$x^{\nu} \in X_{\nu}(x^{-\nu})$$

which depends on the decision variables of the other players.

Aim of the player ν, given the strategy $x^{-\nu}$, is to choose a strategy x^ν such that x^ν solves the following optimization problem

$$(P_\nu) \quad \min_{x^\nu} \theta_\nu(x^\nu, x^{-\nu}), \quad \text{subject to} \quad x^\nu \in X_\nu(x^{-\nu}),$$

where $\theta_\nu(\cdot, x^{-\nu}) : \mathbb{R}^\nu \to \mathbb{R}$ is the decision function for player ν.

In fact, $\theta_\nu(x^\nu, x^{-\nu})$ denotes the loss the player ν suffers when the rival players have chosen the strategy $x^{-\nu}$.

For any given strategy vector $x^{-\nu}$ of the rival players the solution set of the problem (P_ν) is denoted by $S_\nu(x^{-\nu})$.

Thus a vector $\bar{x}$ is a solution of the Generalized Nash Equilibrium if

$$\text{for any } \nu, \quad \bar{x}^\nu \in S_\nu(\bar{x}^{-\nu}).$$

A particular case

Whenever the strategy set of each player does not depend on the choice of the rival players, that is,

$$\text{for any } \nu, \quad X_\nu(x^{-\nu}) \text{ is constant } := X_\nu$$

then the noncooperative game reduces to

$$\text{Find } \bar{x} \in \prod_\nu X_\nu \text{ such that}$$

$$\begin{array}{rcll} \forall\, \nu, \quad \theta_\nu(\bar{x}^\nu, \bar{x}^{-\nu}) & = & \min & \theta_\nu(u, x^{-\nu}) \\ & & \text{s.t.} & u \in X_\nu \end{array}$$

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Find $\bar{x} \in \prod_\nu X_\nu$ such that

$$\forall\, \nu, \quad \theta_\nu(\bar{x}^\nu, \bar{x}^{-\nu}) \;=\; \begin{array}{ll} \min & \theta_\nu(u, x^{-\nu}) \\ \text{s.t.} & u \in X_\nu \end{array}$$

that is, a Nash game.

II

Motivation

Power allocation in telecommunication

Consider a DSL network (Digital Subscriber Line)

- DSL customers connected to the central by dedicaced lines
- wires are bundled together in telephone cables
- electromagnetic coupling $\Rightarrow$ degradation of quality

<u>control variables:</u>

for each wire q and each subcarrier k, p_k^q= power allocated for transmission

<u>constraints:</u>

for each wire q: maximum achievable transmission rate R_q (transmission quality)

It depends of $(p_k)_{k=1,N}$ (power allocations across available subcarriers for q) and of $p^{-q} = (p^r)_{r \neq q}$ (strategies of the other wires)

Power allocation in telecommunication

control variables:

$$R_q(p^q, p^{-q}) = \sum_{k=1}^{N} log(1 + sinr_k^q)$$

where

$$sinr_k^q = \frac{|H_k^{qq}|^2.p_k}{\sigma_q^2}(k) + \sum_{r \neq q} |H_k^{qr}|.p_k^r.$$

(Signal-t-Interference Noise Ratio)

Garantee of minimal transmission rate $R_q(p^q, p^{-q}) \geq R_q^*$.

Power allocation in telecommunication

Model:

each wire wants is a player of the game whose objective is to minimize to total power used for transmission, with the constraint that the maximum transmission rate is at least R_q^*, that is

$$\text{for any } q, \quad \text{solve } (P_q) \qquad \begin{array}{ll} \min & \sum_k = 1^k p_k^q \\ s.t. & \left\{ \begin{array}{l} R_q(p^q, p^{-q}) \geq R_q^* \\ p_k^q \geq 0 \end{array} \right. \end{array}$$

II

Reformulations of GNEP

a- Classical reformulations

Suppose that for any ν and any $x^{-\nu} \in \mathbb{R}^{n^{-\nu}}$, function $\theta_\nu(\cdot, x^{-\nu})$ is continuously differentiable and convex and $X_\nu(x^{-\nu})$ is convex.

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Denoting by

$$X(x) = \prod_{\nu} X_\nu(x^{-\nu}), \quad \forall\, x \in \mathbb{R}^n$$

and

$$F(x) = (\nabla_{x^1}\theta_1(x), \ldots, \nabla_{x^N}\theta_N(x)) \quad \in \mathbb{R}^n$$

we have the reformulation

$$\bar{x} \text{ gene. Nash equil. } \Leftrightarrow \begin{cases} \bar{x} \in X(\bar{x}) \text{ and} \\ \langle F(\bar{x}), y - \bar{x} \rangle \geq 0, \quad \forall\, y \in X(\bar{x}) \end{cases}$$

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that is a quasi-variational inequality.

The jointly convex case

Let us consider a special form of the sets $X_\nu(x^{-\nu})$. This form was originally used by Rosen in his fundamental paper (1965):

Given a nonempty convex subset X of $\mathbb{R}^n$, for any ν, the set $X_\nu(x^{-\nu})$ is given as

$$X_\nu(x^{-\nu}) = \{x^\nu \in \mathbb{R}^{n_\nu} : (x^\nu, x^{-\nu}) \in X\}.$$

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Define the Nikaido-Isoda (or Ky Fan) function $\Psi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$\Psi(x, y) = \sum_{i=1}^{N} \left[\theta_\nu(x^\nu, x^{-\nu}) - \theta_\nu(y^\nu, x^{-\nu})\right]$$

Reformulation by V.I.

Then the GNEP can be reformulated in the following form

$$\min_{x \in X} V(x)$$

where $V(x) = \sup_{y \in X} \Psi(x, y)$.

Proposition 1 *If the decision functions θ_ν are C^1, then any solution of the Stampacchia variational inequality defined by X and*

$$F(x) = (\nabla_{x^1}\theta_1(x), \ldots, \nabla_{x^N}\theta_N(x)) \quad \in \mathbb{R}^n$$

is a solution of the GNEP.

b- An extended reformulation

To simplify the notations, we will denote, for any ν and any $x \in \mathbb{R}^n$, by $S_\nu(x)$ and $A_\nu(x^{-\nu})$ the subsets of $\mathbb{R}^{n_\nu}$

$$S_\nu(x) = S^a_{\theta_\nu(\cdot,x^{-\nu})}(x^\nu) \quad \text{and} \quad A_\nu(x^{-\nu}) = \arg\min_{\mathbb{R}^{n_\nu}} \theta_\nu(\cdot, x^{-\nu}).$$

In order to construct the variational inequality problem we define the following set-valued map $N^a_\theta : \mathbb{R}^n \longrightarrow 2^{\mathbb{R}^n}$ which is described, for any $x = (x^1, \ldots, x^p) \in \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_p}$, by

$$N^a_\theta(x) = F_1(x) \times \ldots \times F_p(x),$$

$$\text{where } F_\nu(x) = \begin{cases} \overline{B}_\nu(0,1) & \text{if } x^\nu \in A_\nu(x^{-\nu}) \\ \text{co}(N^a_{\theta_\nu}(x^\nu) \cap S_\nu(0,1)) & \text{otherwise} \end{cases}$$

The set-valued map N^a_θ has nonempty convex compact values.

Lemma 2 *Let $\nu \in \{1, \ldots, p\}$. If the function θ_ν is continuous quasiconvex with respect to the ν-th variable, then,*

$$0 \in F_\nu(\bar{x}) \iff \bar{x}^\nu \in A_\nu(\bar{x}^{-\nu}).$$

Proof. It is sufficient to consider the case of a point $\bar{x}$ such that $\bar{x}^\nu \notin A_\nu(\bar{x}^{-\nu})$. Since $\theta_\nu(\cdot, \bar{x}^{-\nu})$ is continuous at $\bar{x}^\nu$, the interior of $S_\nu(\bar{x})$ is nonempty. Let us denote by K_ν the convex cone

$$K_\nu = N^a_{\theta_\nu}(\bar{x}^\nu) = (S_\nu(\bar{x}) - \bar{x}^\nu)^\circ.$$

By quasiconvexity of θ_ν, K_ν is not reduced to $\{0\}$. Let us first observe that, since $S_\nu(\bar{x})$ has a nonempty interior, K_ν is a pointed cone, that is $K_\nu \cap (-K_\nu) = \{0\}$.
Now let us suppose that $0 \in F_\nu(\bar{x})$. By Caratheodory theorem, there exist vectors $v_i \in [K_\nu \cap S_\nu(0,1)]$, $i = 1, \ldots, n+1$ and scalars $\lambda_i \geq 0$, $i = 1, \ldots, n+1$ with

$$\sum_{i=1}^{n+1} \lambda_i = 1 \text{ and } 0 = \sum_{i=1}^{n+1} \lambda_i \, v_i.$$

Since there exists at least one $r \in \{1, \ldots, n+1\}$ such that $\lambda_r > 0$ we have

$$v_r = -\sum_{i=1, i \neq r}^{n+1} \frac{\lambda_i}{\lambda_r} v_i$$

which clearly shows that v_r is an element of the convex cone $-K_\nu$. But $v_r \in S_\nu(0,1)$ and thus $v_r \neq 0$. This contradicts the fact that K_ν is pointed and the proof is complete. ■

Sufficient condition

In the following we assume that X is a given nonempty subset X of $\mathbb{R}^n$, such that for any ν, the set $X_\nu(x^{-\nu})$ is given as

$$X_\nu(x^{-\nu}) = \{x^\nu \in \mathbb{R}^{n_\nu} : (x^\nu, x^{-\nu}) \in X\}.$$

Theorem 4 *Let us assume that, for any ν, the function θ_ν is continuous and quasiconvex with respect to the ν-th variable. Then every solution of $VI(N_\theta^a, X)$ is a solution of the GNEP.*

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Theorem 5 *Let us assume that, for any ν, the function θ_ν is continuous and quasiconvex with respect to the ν-th variable. Then every solution of $VI(N_\theta^a, X)$ is a solution of the GNEP.*

Note that the link between GNEP and variational inequality is valid even if the constraint set X is neither convex nor compact.

Sufficient optimality condition

Proposition 6

$f : X \to \mathbb{R} \cup \{+\infty\}$ *quasiconvex, radially cont. on* dom f

$C \subseteq X$ *such that* $conv(C) \subset$ dom f.

Suppose that $C \subset int($dom $f)$.

Then $\bar{x} \in S(N_f^a \setminus \{0\}, C) \quad \Longrightarrow \quad \forall\, x \in C, f(\bar{x}) \leq f(x)$.

where $\bar{x} \in S(N_f^a \setminus \{0\}, K)$ means that there exists $\bar{x}^* \in N_f^a(\bar{x}) \setminus \{0\}$ such that

$$\langle \bar{x}^*, c - x \rangle \geq 0, \qquad \forall\, c \in C.$$

Proof. Let us consider $\bar{x}$ to be a solution of $VI(N_\theta^a, X)$. There exists $v \in N_\theta^a(\bar{x})$ such that

$$\langle v, y - \bar{x} \rangle \geq 0, \quad \forall\, y \in X. \qquad (*)$$

Let $\nu \in \{1, \ldots, p\}$.
If $\bar{x}^\nu \in A_\nu(\bar{x}^{-\nu})$ then obviously $\bar{x}^\nu \in Sol_\nu(\bar{x}^{-\nu})$.
Otherwise $v^\nu \in F_\nu(\bar{x}) = \mathrm{co}(N_{\theta_\nu}^a(\bar{x}^\nu) \cap S_\nu(0,1))$. Thus, according to Lemma 2, there exist $\lambda > 0$ and $u^\nu \in N_{\theta_\nu}^a(\bar{x}^\nu) \setminus \{0\}$ satisfying $v^\nu = \lambda u^\nu$.
Now for any $x^\nu \in X_\nu(\bar{x}^{-\nu})$, consider $y = \left(\bar{x}^1, \ldots, \bar{x}^{\nu-1}, x^\nu, \bar{x}^{\nu+1}, \ldots, \bar{x}^p\right)$.
From $(*)$ one immediately obtains that $\langle u^\nu, x^\nu - \bar{x}^\nu \rangle \geq 0$. Since x^ν is an arbitrary element of $X_\nu(\bar{x}^{-\nu})$, we have that $\bar{x}^\nu$ is a solution of $VI(N_{\theta_\nu}^a \setminus \{0\}, X_\nu(\bar{x}^{-\nu}))$ and therefore, according to Prop. 4,

$$\bar{x}^\nu \in Sol_\nu(\bar{x}^{-\nu})$$

Since ν was arbitrarily chosen we conclude that $\bar{x}$ solves the GNEP. ■

Necessary and sufficient condition

Theorem 7 *Let us suppose that, for any ν, the loss function θ_ν is continuous and semistrictly quasiconvex with respect to the ν-th variable. Further assume that the set X is a nonempty convex subset of $\mathbb{R}^n$.*
Then $\bar{x}$ be a solution of the GNEP if and only if $\bar{x}$ is a solution of the variational inequality $VI(N_\theta^a, X)$.

Existence result 1

Theorem 8 (Ichiishi 83) *Assume that for every player* ν*, the loss function* θ_ν *is continuous on* $\mathbb{R}^n$ *and quasiconvex with respect to the* ν*-th variable. If the set-valued map is continuous with nonempty convex compact vales, then the generalized Nash equilibrium problem admits a solution.*

Existence result 2

Theorem 9 *Assume that for every player ν, the loss function θ_ν is continuous on $\mathbb{R}^n$ and semistrictly quasiconvex with respect to the ν-th variable. If the set X is nonempty, convex and compact, then the generalized Nash equilibrium problem admits a solution.*

IV

Special important cases

In case of a unique leader, the game can be reformulated as bilevel programming problem:

$$(BL) \qquad \inf \quad f_u(x,y) \qquad \text{s.t.} \begin{cases} y \in S(x) \\ g_k(x,y) \leq 0, \;\; k = 1,\ldots,p \end{cases}$$

where $S(x)$ is the solution set of the *lower level problem*

$$(PL_x) \qquad \inf_{y'} \quad f_l(x,y') \qquad \text{s.t.} \begin{cases} (x,y') \in C \\ h_j(x,y') \leq 0, \;\; j = 1,\ldots,q \end{cases}$$

Let us consider the set-valued map $S : X \to 2^Y$ which associates to any point x the (possibly empty) solution set of the lower level problem (PL_x) defined by x, that is

$$S(x) = \arg\min_{\Omega(x)} f_l(x, \cdot)$$

where $\Omega(x)$ is the feasible region of the lower level problem (PL_x), namely

$$\Omega(x) = \{(x, y) \in X \times Y \ : \ (x, y) \in C \text{ and } h_j(x, y) \leq 0, \ \ j = 1, \ldots, q\}.$$

Existence for bilevel

Theorem 11 *Let* $f_u : X \times Y \to \mathbb{R} \cup \{+\infty\}$ *be a lsc quasiconvex function, radially continuous on* $\operatorname{dom} f_u$. *Assume that*

a) *for every* $\lambda > \inf_{X \times Y} f_u$, $\operatorname{int}(S_\lambda(f_u)) \neq \emptyset$;

b) $\operatorname{Gr}(S)$ *is a locally finite union of a family* $\{K_\alpha : \alpha \in A\}$ *of closed convex sets of* $X \times Y$;

c) *the functions* g_k *(*$k = 1, \ldots, p$*) are lsc quasiconvex on* $X \times Y$;

d) *one of the following assumption holds:*

  i) *for any* $\alpha \in A$, K_α *is weakly compact and bounded;*

  ii) *the set* $M = \{(x, y) \,:\, g_k(x, y) \leq 0, \;\; k = 1, \ldots, p\}$ *is weakly compact and bounded;*

e) *the feasible region of problem* (BL) *is nonempty.*

Then if A *is finite the bilevel programming problem* (BL) *admits a global solution;*

Existence for bilevel

Theorem 12 *Let $f_u : X \times Y \to \mathbb{R} \cup \{+\infty\}$ be a lsc quasiconvex function, radially continuous on* $\mathrm{dom}\, f_u$. *Assume that*

a) *for every $\lambda > \inf_{X \times Y} f_u$, $\mathrm{int}(S_\lambda(f_u)) \neq \emptyset$;*

b) $\mathrm{Gr}\,(S)$ *is a locally finite union of a family $\{K_\alpha : \alpha \in A\}$ of closed convex sets of $X \times Y$;*

c) *the functions g_k ($k = 1, \ldots, p$) are lsc quasiconvex on $X \times Y$;*

d) *one of the following assumption holds:*

  i) *for any $\alpha \in A$, K_α is weakly compact and bounded;*

  ii) *the set $M = \{(x, y) \,:\, g_k(x, y) \leq 0, \;\; k = 1, \ldots, p\}$ is weakly compact and bounded;*

e) *the feasible region of problem (BL) is nonempty.*

Then if A is not finite but there exists a local mapping $\mathcal{M} = \{(\rho_w, A_w) \,:\, w \in \mathrm{Gr}\,(S)\}$ of $\mathrm{Gr}\,(S)$ *such that the set $\{w \in \mathrm{Gr}\,(S) \,:\, card(A_w) > 1\}$ is included in a weakly compact subset of* $\mathrm{Gr}\,(S)$, *the bilevel programming problem (BL) admits a local solution.*

The linear Bilevel problem

Let us consider the following particular case of the bilevel problem

$$(BL\_Lin) \qquad \inf \quad f_u(x,y)$$
$$\text{s.t.} \quad \begin{cases} y \in S(x) \\ L_1(x,y) \leq 0 \end{cases}$$

where $S(x)$ is the solution set of the linear lower level problem

$$(PL_x\_Lin) \qquad \inf_{y'} \quad L_2(x,y')$$
$$\text{s.t.} \quad L_3(x,y') \leq 0$$

where L_1, L_2 and L_3 are linear continuous functions defined on $X \times Y$ with, respectively, values in $\mathbb{R}^p$, $\mathbb{R}$ and $\mathbb{R}^q$.

Existence for linear Bilevel problem

Corollary 13 *Let* $f_u : X \times Y \to \mathbb{R} \cup \{+\infty\}$ *be a lower semicontinuous quasiconvex function, radially continuous on* $\text{dom}\, f_u$. *Assume that*

a) *for every* $\lambda > \inf_{X \times Y} f_u$, $\text{int}(S_\lambda(f_u)) \neq \emptyset$;

c) *the functions* L_1, L_2 *and* L_3 *are linear continuous functions;*

d) *the set* $M = \{(x, y) \,:\, L_3(x, y) \leq 0, \;\; k = 1, \ldots, p\}$ *is weakly compact and bounded;*

e) *the feasible region of problem* (BL) *is nonempty.*

Then the bilevel programming problem (BL_Lin) *admits a global solution.*

Another case

For any $\alpha \in A$, the marginal function $l_\alpha : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ of the lower level subproblem on C_α will be defined by

$$l_\alpha(x) = \inf_{y'} \quad f_l(x, y')$$
$$\text{s.t.} \quad \begin{cases} (x, y') \in C_\alpha \\ h_j(x, y') \leq 0, \quad j = 1, \ldots, q \end{cases}$$

Theorem 15 *Let $f_u : X \times Y \to \mathbb{R} \cup \{+\infty\}$ be a lsc quasiconvex function, radially continuous on* dom f_u *and $f_l : X \times Y \to \mathbb{R} \cup \{+\infty\}$ be a lsc quasiconvex function. Assume that*

- *a) for every $\lambda > \inf_{X \times Y} f_u$, $\text{int}(S_\lambda(f_u)) \neq \emptyset$;*
- *b) C is a locally finite union of a family $\{C_\alpha : \alpha \in A\}$ of closed convex sets of $X \times Y$;*
- *c) the functions g_k ($k = 1, \ldots, p$) and h_j ($j = 1, \ldots, q$) are lower semicontinuous quasiconvex on $X \times Y$;*
- *d) for any $\alpha \in A$, C_α is weakly compact and bounded;*
- *e) for any $\alpha \in A$, for any x, x' such that $(\{x\} \times Y) \cap C_\alpha \neq \emptyset$ and $(\{x'\} \times Y) \cap C_\alpha \neq \emptyset$, $l_\alpha(x) = l_\alpha(x')$;*
- *f) the feasible region of problem (BL) is nonempty.*

Then if A is finite the bilevel programming problem (BL) admits a global solution;

Theorem 16 *Let* $f_u : X \times Y \to \mathbb{R} \cup \{+\infty\}$ *be a lsc quasiconvex function, radially continuous on* $\operatorname{dom} f_u$ *and* $f_l : X \times Y \to \mathbb{R} \cup \{+\infty\}$ *be a lsc quasiconvex function. Assume that*

a) *for every* $\lambda > \inf_{X \times Y} f_u$, $\operatorname{int}(S_\lambda(f_u)) \neq \emptyset$;

b) C *is a locally finite union of a family* $\{C_\alpha : \alpha \in A\}$ *of closed convex sets of* $X \times Y$;

c) *the functions* g_k $(k = 1, \ldots, p)$ *and* h_j $(j = 1, \ldots, q)$ *are lower semicontinuous quasiconvex on* $X \times Y$;

d) *for any* $\alpha \in A$, C_α *is weakly compact and bounded;*

e) *for any* $\alpha \in A$, *for any* x, x' *such that* $(\{x\} \times Y) \cap C_\alpha \neq \emptyset$ *and* $(\{x'\} \times Y) \cap C_\alpha \neq \emptyset$, $l_\alpha(x) = l_\alpha(x')$;

f) *the feasible region of problem* (BL) *is nonempty.*

Then if A *is infinite but there exists a local mapping* $\mathcal{M} = \{(\rho_{(x,y)}, A_{(x,y)}) : (x, y) \in C\}$ *of* C *such that the set* $\{(x, y) \in C : card(A_{(x,y)}) > 1\}$ *is included in a weakly compact subset of* C, *then the bilevel programming problem* (BL) *admits a local solution.*

Convex differentiable game with unique leader

If there is a unique leader and the decision functions θ_ν are differentiable and convex, writing optimality conditions, the GNEP can be reformulated as a Mathematical Programming with Equilibrium constraints (MPEC)

Convex differentiable game with unique leader

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$$\text{(MPEC)} \qquad \inf f(z)$$
$$\text{s. t.} \quad \begin{cases} g(z) \le 0 \\ h(z) = 0 \\ G(z) \ge 0, H(z) \ge 0 \\ \langle G(z), H(z) \rangle = 0 \end{cases}$$

An existence result for MPEC

$$\text{(MPEC)} \qquad \inf f(z) \qquad \text{s. t.} \quad \begin{cases} g(z) \leq 0 \\ h(z) = 0 \\ G(z) \geq 0, H(z) \geq 0 \\ \langle G(z), H(z) \rangle = 0 \end{cases}$$

$f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$, $g : X \longrightarrow \mathbb{R}^p$ quasiconvex
$h : X \longrightarrow \mathbb{R}^q$ and $G, H : X \longrightarrow \mathbb{R}^m$ quasiaffine (i.e. each coordinate function is quasiaffine)

Proposition 17

$f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ *quasiconvex*
+ *lsc, radial* ^{ly} *continuous on* dom (f)
+ *for any* $\lambda > \inf_X f$, *int*$(S_\lambda) \neq \emptyset$.
+ $C \subseteq$ *int*(dom f)
+ g *quasiconvex continuous*
+ h, H *and* G *quasiaffine continuous*
+ *coercivity condition*

Then the Quasiconvex-quasiaffine MPEC problem admits a global minimizer.

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