Overview on Generalized Convexity and Vector Optimization

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Contents

$E \neq \emptyset$ with partial order (reflexive and transitive) $\preccurlyeq$; $A \subseteq E$. $\bar{a} \in A$ is efficient of A if

$$a \in A\,, a \preccurlyeq \bar{a} \Longrightarrow \bar{a} \preccurlyeq a.$$

The set of $\bar{a}$ is denoted $\mathrm{Min}(A, \preccurlyeq)$. Given $x \in E$, lower and upper section at x,

$$L_x \doteq \{y \in E :\ y \preccurlyeq x\},\ S_x \doteq \{y \in E :\ x \preccurlyeq y\},$$

Set

$$S_A \doteq \bigcup_{x \in A} S_x.$$

When $\preccurlyeq = \leq_P$, P being a convex cone, then

$$(x \preccurlyeq y \Longleftrightarrow y - x \in P)\ L_x = x - P,\ S_x = x + P,\ S_A = A + P.$$

- Property (Z): each totally ordered (chain) subset of A has a lower bound in A

$$\Updownarrow$$

- A is order-totally-complete (it has no covering of form $\{(L_x)^c: \ x \in D\}$ with $D \subseteq A$ being totally ordered)

$$\Updownarrow$$

- each maximal totally ordered subset of A has a lower bound in A.

$$A \not\subset \bigcup_{x \in D} L_x^c \Leftrightarrow \emptyset \neq A \cap \left(X \setminus \bigcup_{x \in D} L_x^c \right) \Leftrightarrow \emptyset \neq A \bigcap \bigcap_{x \in D} L_x \Leftrightarrow \exists \, \mathrm{LB}.$$

Sonntag-Zalinescu, 2000; Ng-Zheng, 2002; Corley, 1987; Luc, 1989; Ferro, 1996, 1997, among others.

Basic Definitions:

(a) [Ng-Zheng, 2002] A is order-semicompact (resp. order-s-semicompact) if every covering of A of form $\{L_x^c : \ x \in D\}$, $D \subseteq A$ (resp. $D \subseteq E$), has a finite subcover.

(b) [Luc, 1989; FB-Hernández-Novo, 2008] A es order-complete if $\nexists$ covering of form $\{L_{x_\alpha}^c : \alpha \in I\}$ where $\{x_\alpha : \alpha \in I\}$ is a decreasing net in A.

A directed set $(I, >)$ is a set $I \neq \emptyset$ together with a reflexive and transitive relation $>$: for any two elements $\alpha, \ \beta \in I$ there exists $\gamma \in I$ with $\gamma > \alpha$ and $\gamma > \beta$.
A net in E is a map from a directed set $(I, >)$ to E. A net $\{y_\alpha : \ \alpha \in I\}$ is *decreasing* if $y_\beta \preccurlyeq y_\alpha$ for each $\alpha, \beta \in I$, $\beta > \alpha$.

Theorem

If A is order-totally-complete then $\mathrm{Min}\, A \neq \emptyset$.

Proof. Let $\mathcal{P}$ = set of totally ordered sets in A. Since $A \neq \emptyset$, $\mathcal{P} \neq \emptyset$. Moreover, $\mathcal{P}$ equipped with the partial order - inclusion, becomes a partially ordered set. By standard arguments we can prove that any chain in $\mathcal{P}$ has an upper bound and, by Zorn's lemma, we get a maximal set $D \in \mathcal{P}$.
Applying a previous equivalence, there exists a lower bound $a \in A$ of D. We claim that $a \in \mathrm{Min}\, A$. Indeed, if $a' \in A$ satisfies that $a' \preccurlyeq a$ then a' is also a lower bounded of D. Thus, $a' \in D$ by the maximality of D in $\mathcal{P}$. Hence, $a \preccurlyeq a'$ and therefore $a \in \mathrm{Min}\, A$.
In particular, if $A \subseteq E$ is order-s-semicompact, order-semicompact or order-complete, then $\mathrm{Min}\, A \neq \emptyset$.

Teorema [Ng-Zheng, 2002; FB-Hernández-Novo, 2008]

The following are equivalent:

(*a*) $\mathrm{Min}(A, \preccurlyeq) \neq \emptyset$;

(*b*) A has a maximal totally ordered subset minorized by an order-s-semicompact subset H of S_A;

(*c*) A has a nonempty section which is order-complete;

(*d*) A has a nonempty section which is order-totally-complete (equiv. satisfies property (Z)).

$$S_A \doteq \bigcup_{x \in A} \{y \in E : \; x \preccurlyeq y\}.$$

$$(\preccurlyeq = \leq_P, \; l(P) = \{0\}); \bar{a} \in \mathrm{Min}\, A \Longleftrightarrow A \cap (\bar{a} - P) = \{\bar{a}\}.$$

Sketch - proof

$(a) \Longrightarrow (b)$: Take $a \in \mathrm{Min}\, A$, and consider

$$\mathcal{P} \doteq \{D \subseteq E \colon L_a \cap A \subseteq D \subseteq S_a \cap A \text{ and } D \text{ is totally ordered }\}.$$

It is clear $L_a \cap A$ is totally ordered, $L_a \cap A \in \mathcal{P}$. By equipping $\mathcal{P}$ with the partial order - inclusion- we can prove by standard arguments that any chain in $\mathcal{P}$ has an upper bound. Therefore, there exists a maximal totally ordered element $D_0 \in \mathcal{P}$, i.e.,

$$L_a \cap A \subseteq D_0 \subseteq S_a \cap A \subseteq S_a.$$

Set $H = \{a\}$. Then D_0 is minorized by H which is an order-s-semicompact subset of S_A.
It generalizes and unifies results by Luc 1989, Ng-Zheng 2002 among others.

Optimization problem

X Hausdorff top. s.p; $f : X \to (E, \preccurlyeq)$. Consider

$$\min\{f(x) : \; x \in X\} \qquad (P)$$

$f(X) \doteq \{f(x) : \; x \in X\}$. A sol $\bar{x} \in X$ to (P) is such that $f(\bar{x}) \in \mathrm{Min}(f(X), \preccurlyeq)$.

Theorem [FB-Hernández-Novo, 2008]

Let X compact. If $f^{-1}(L_y)$ closed $\forall\, y \in f(X)$ (resp. $\forall\, y \in E$), then $f(X)$

- (a) is order-semicomp. (resp. $f(X)$ is order-s-semicomp.);
- (b) has the domination property, i.e., every lower section of $f(X)$ has an efficient point.

As a consequence, $\mathrm{Min}(f(X), \preccurlyeq) \neq \emptyset$.

Proof.

We only prove (a) when $f^{-1}(L_y)$ is closed for all $y \in f(X)$. Suppose $\bigcup_{d\in D} L_d^c$ is a covering of $f(X)$ with$D \subseteq f(X)$. Put

$$U_d \doteq \{x \in X \colon f(x) \notin L_d\}.$$

Then, $X = \bigcup_{d\in D} U_d$. Since $f^{-1}(L_d)$ is closed, $U_d = (f^{-1}(L_d))^c$ is open $\forall\, d \in D$. Moreover, as X is compact, $\exists$ finite set $\{d_1, \ldots, d_r\} \subseteq D$ such that

$$X = U_{d_1} \cup \cdots \cup U_{d_r}.$$

Hence, $L_{d_1}^c \cup \cdots \cup L_{d_r}^c$ covers $f(X)$ and therefore $f(X)$ is order-semicompact.

We introduce the following new

Definition [FB-Hernández-Novo, 2008]: Let $x_0 \in X$.

We say f is decreasingly lower bounded at x_0 if for each net $\{x_\alpha : \alpha \in I\}$ convergent to x_0 such that $\{f(x_\alpha) : \alpha \in I\}$ is decreasing, the following holds

$$\forall\, \alpha \in I: \quad f(x_0) \in L_{f(x_\alpha)}.$$

We say that f is decreasingly lower bounded (in X) if it is for each $x_0 \in X$.

Proposition [FB-Hernández-Novo, 2008]

If $f^{-1}(L_y)$ is closed $\forall\, y \in f(X)$, then f is decreasingly lower bounded.

Theorem [FB-Hernández-Novo, 2008]

Let X compact. If f is decreasingly lower bounded, then

(a) $f(X)$ is order-complete;

(b) $f(X)$ has the domination property;

(c) $\mathrm{Min} f(X) \neq \emptyset$.

Special situation

Y top. vec. space ordered by a closed convex cone $P \subseteq Y$.
Define $\preccurlyeq^l$ in 2^Y. If $A, B \in 2^Y$ then

$$A \preccurlyeq^l B \iff B \subseteq A + P.$$

This is partial order: reflexive and transitive [Jahn, 2003; Kuroiwa, 1998, 2003].
Kuroiwa introduces the notion of efficient set for a family of $\mathcal{F} \subseteq$ of nonempty subsets of Y. We say $A \in \mathcal{F}$ is a l-minimal set ($A \in \mathrm{lMin}\mathcal{F}$) if

$$B \in \mathcal{F},\ B \preccurlyeq^l A \implies A \preccurlyeq^l B.$$

- X real Hausd. top. vect. spac.; Y real normed vect. spac.;
- $P \subseteq Y$ a convex cone, $\text{int}\, P \neq \emptyset$, $l(P) \doteq P \cap (-P)$;
- $K \subseteq X$ a closed set; $F : K \to Y$ a vector function.

E = the set of $\bar{x}$ such that

$$\bar{x} \in K : \; F(x) - F(\bar{x}) \notin -P \setminus l(P) \;\; \forall\, x \in K.$$

Its elements are called efficient points;
E_W = the set of $\bar{x}$ such that

$$\bar{x} \in K : \; F(x) - F(\bar{x}) \notin -\text{int}\, P \;\; \forall\, x \in K.$$

Its elements are called weakly efficient points.

$$E \subseteq E_W = \bigcap_{x \in K} \Big\{ \bar{x} \in K : \; F(\bar{x}) - F(x) \notin \text{int}\, P \Big\}.$$

How we can compute the efficient points?

Theorem: Consider $P = \mathbb{R}^n_+$, $F(x) = Cx$ (linear), K polyhedra.

$\bar{x}$ is efficient $\Longleftrightarrow \exists\ p^* > 0$ such that $\bar{x}$ solves

$$\min\{\langle p^*, F(x)\rangle :\ Ax \geq b,\ x \geq 0\}.$$

In a standar notation $\bar{x} \in \operatorname{argmin}_K \langle p^*, F(\cdot)\rangle$,
$K \doteq \{x \in \mathbb{R}^n :\ Ax \geq b,\ x \geq 0\}$.
Does the previous theorem remains valid for non linear F?

$$\bar{x} \in E \Longleftrightarrow \bar{x} \in \bigcup_{p^* \in \mathbb{R}^m_{++}} \operatorname{argmin}_K \langle p^*, F(\cdot)\rangle \ (\Longleftarrow \ \text{always!!});$$

$\Longrightarrow$ weighting method

How to choice $p^* \in \mathbb{R}^m_{++}$?

Example 1.1.

Let $F(x_1, x_2) = (x_1, x_2)$, $x \in K \doteq \{(x_1, x_2) \in \mathbb{R}^2 : \; x_1 + x_2 \geq 1\}$. Here, $E = E_W = \{(x_1, x_2) \in \mathbb{R}^2 : \; x_1 + x_2 = 1\}$. However,

$$\inf_{x \in \mathbb{R}} \langle p^*, F(x) \rangle = -\infty, \; p^* = (p_1^*, p_2^*), \; p_1^* \neq p_2^*.$$

Example 1.2.

Let $F(x) = (\sqrt{1 + x^2}, x)$, $x \in K = \mathbb{R}$. Here, $E = E_W = \;]-\infty, 0]$. However, if $p_2^* > p_1^* > 0$, and

$$\inf_{x \in \mathbb{R}} \langle p^*, F(x) \rangle = -\infty, \; p^* = (p_1^*, p_2^*).$$

A lot of work to do !!!

Let $h : X \to \mathbb{R} \cup \{+\infty\}$. It is

- quasiconvex if

$$h(x) \leq h(y) \Longrightarrow h(\xi) \leq h(y) \quad \forall \xi \in (x, y);$$

  or equivalently, $\{x : \; h(x) \leq t\}$ is convex for all $t \in \mathbb{R}$.

- semistrictly quasiconvex if

$$h(x) < h(y) \Longrightarrow h(\xi) < h(y) \quad \forall \, \xi \in (x, y).$$

Proposition

If $h : X \to \mathbb{R} \cup \{+\infty\}$ is semistrictly quasiconvex and lower semicontinuous, then it is quasiconvex.

Theorem [Malivert-Boissard, 1994] $K \subseteq \mathbb{R}^n$ convex

each f_i ($i = 1, \ldots, m$) is quasiconvex, semistrictly quasiconvex, and lsc along lines in K. Then

$$E_W = \bigcup \{E(J) : \; J \subseteq \{1, \ldots, m\}, J \neq \emptyset\}.$$

Example 2.

Consider $F = (f_1, f_2)$, $K = [0, +\infty[$,

$$f_1(x) = \begin{cases} 2, & \text{if } x \notin [1,2] \\ 1, & \text{if } x \in [1,2] \end{cases} \qquad f_2(x) = |x - 5|.$$

Here, $E = \{2, 5\}$, $E_W = [1, 8]$.

Some notations

$$G^\lambda \doteq \{x \in K : F(x) - \lambda \in -P\}; G_\lambda \doteq \{x \in K : \ F(x) - \lambda \notin \mathrm{int}\, P\};$$

$$G(y) \doteq \{x \in K : \ F(x) - F(y) \notin \mathrm{int}\, P\};$$

$$\mathrm{epi}\, F \doteq \{(x, y) \in K \times Y : \ y \in F(x) + P\}.$$

There is no relationship between the closedness of G^λ for all $\lambda \in Y$ and the closedness of $G(y)$ for all $y \in K$ even when P is additionally closed.

- $F : K \to Y$ is [Penot-Therá, 1979] P-lower semicontinuous (P-lsc) at $x_0 \in K$ if $\forall$ open set $V \subseteq Y$ st $F(x_0) \in V$ $\exists$ an open neighborhood $U \subseteq X$ of x_0 st $F(U \cap K) \subseteq V + P$. We shall say that F is P-lsc (on K) if it is at every $x_0 \in K$.
- F is $\mathbb{R}^m_+$-lsc if and only if each f_i is lsc.

Proposition [FB, 2003; Bianchi-Hadjisavvas-Schaible, 1997; The Luc, 1989]

$P \subseteq Y$ is convex cone, $K \subseteq X$ and $S \subseteq Y$ be closed sets such that $S + P \subseteq S$ and $S \neq Y$; $F : K \to Y$. The following hold.

(a) If F is a P-lsc function, then $\{x \in K : \ F(x) \in \lambda - S\}$ is closed for all $\lambda \in Y$;

(b) Assume $\operatorname{int} P \neq \emptyset$ and P closed: F is P-lsc if and only if $\{x \in K : \ F(x) - \lambda \notin \operatorname{int} P\}$ is closed for all $\lambda \in Y$;

(c) Assume $\operatorname{int} P \neq \emptyset$ and P closed: $\operatorname{epi} F$ is closed if and only if $\{x \in K : \ F(x) - \lambda \in -P\}$ is closed for all $\lambda \in Y$;

(d) Assume $\operatorname{int} P \neq \emptyset$ and P closed: if F is P-lsc then $\operatorname{epi} F$ is closed.

Theorem [Ferro, 1982; (set-valued) Ng-Zheng, 2002]

P convex cone; K compact; G^λ closed for all $\lambda \in Y$ ($\Longleftrightarrow \mathrm{epi}\, F$ is closed if $\mathrm{int}\, P \neq \emptyset$). Then $E \neq \emptyset$.

Proof. We know $\mathrm{Min}\, F(X) \neq \emptyset$, thus $E \neq \emptyset$.

$$G^\lambda \doteq \{x \in K : F(x) - \lambda \in -P\}.$$

Theorem: P convex cone, $\operatorname{int} P \neq \emptyset$; K compact;

$G(y) \doteq \{x \in K : \; F(x) - F(y) \notin \operatorname{int} P\}$ closed $\forall\, y \in K$. Then $E_W \neq \emptyset$.

Proof. Notice that $E_W = E(F(K)|C)$ for $C = (\operatorname{int} P) \cup \{0\}$. The closedness of E_W is obvious; it suffices to show that $E_W \neq \emptyset$. If it is not order-complete for C, let $\{F(x_\alpha)\}$ be a decreasing net with $\{(F(x_\alpha) - C)^c\}_\alpha$ forming a covering of $F(K)$. By compactness, (assume) $x_\alpha \to x_0$ for some $x_0 \in K$. If $E_W = \emptyset$, $\exists$ $y \in K$ such that $F(y) - F(x_0) \in -\operatorname{int} P$. For $F(y)$, $\exists\ \alpha_0$ such that $F(y) - F(x_{\alpha_0}) \notin -C$. This implies

$$F(y) - F(x_\alpha) = F(y) - F(x_{\alpha_0}) + F(x_{\alpha_0}) - F(x_{\alpha_0})$$

$$\in (Y \setminus -C) + C \subseteq Y \setminus -C \subseteq Y \setminus -\operatorname{int} P \;\; \forall\, \alpha > \alpha_0.$$

$G(y)$ closed implies $F(x_0) - F(y) \notin \operatorname{int} P$, a contradiction.

Since $\mathrm{int}\ P \neq \emptyset$, take $\bar{y} \in \mathrm{int}\ P$. Then the set

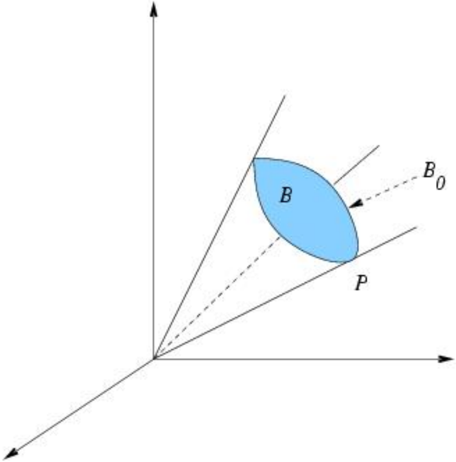
$$B = \{y^* \in P^* : \langle y^*, \bar{y} \rangle = 1\}$$

is a w^*-compact convex base for P^*, i.e., $0 \notin B$ and $P^* = \bigcup_{t \geq 0} tB$. In this case,

$$p \in P \iff \langle p^*, p \rangle \geq 0 \ \ \forall\ p^* \in B;$$

$$p \in \mathrm{int}\ P \iff \langle p^*, p \rangle > 0 \ \ \forall\ p^* \in B.$$

The set E^* of the extreme points of B is nonempty by the Krein-Milman theorem.


B
B_0
P

Definitions: Let $\emptyset \neq K \subseteq X$, $F : K \to Y$ is said to be:

1. P-convex if, $x, y \in K$,

   $$tF(x) + (1-t)F(y) \in F(tx + (1-t)y) + P, \quad \forall\, t \in \,]0,1[;$$

   F is $\mathbb{R}^m_+$-convex if and only if each f_i is convex.
2. properly P-quasiconvex [Ferro, 1982] if, $x, y \in K$, $t \in \,]0,1[$,

   $$F(tx+(1-t)y) \in F(x) - P \;\text{ or }\; F(tx+(1-t)y) \in F(x) + P,$$

   or equivalently, $\{\xi \in K : F(\xi) \notin \lambda + P\}$ is convex $\forall\, \lambda \in Y$.
   $F(x) = (x, -x^2)$, $K = \,]-\infty, 0]$, satisfies 2 but not 1;
   $F(x) = (x^2, -x)$, $K = \mathbb{R}$, satisfies 1 but not 2;

More Definitions

3. naturally P-quasiconvex [Tanaka, 1994] if, $x, y \in K$, $t \in \;]0, 1[$

   $$F(tx+(1-t)y) \in \mu F(x)+(1-\mu)F(y)-P, \text{ for some } \mu \in \; [0, 1],$$

   or equivalently, $F([x, y]) \in \mathrm{co}\{F(x), F(y)\} - P$.
   $F(x) = (x^2, 1 - x^2)$, $K = [0, 1]$, satisfies 3 but not 2 or 1.
4. scalarly P-quasiconvex [Jeyakumar-Oettli-Natividad, 1993] if, for $p^* \in P^* \setminus \{0\}$, $x \in K \mapsto \langle p^*, F(x)\rangle$ is quasiconvex.

Both are equivalent [FB-Hadjisavvas-Vera, 2007] if $\mathrm{int}\, P \neq \emptyset$.

$$\Longrightarrow F(K) + P \text{ is convex.}$$

More Definitions

5. *P*-quasiconvex [Ferro, 1982] if,

$$\{\xi \in K : F(\xi) - \lambda \in -P\} \text{ is convex } \forall\, \lambda \in Y.$$

   F is $\mathbb{R}^m_+$-quasiconvex if and only if each f_i is quasiconvex. [Benoist-Borwein-Popovici, 2003] This is equivalent to: given any $p^* \in E^*$, $x \in K \mapsto \langle p^*, F(x)\rangle$ is quasiconvex.

6. semistrictly-*P*-quasiconvex at y [Jahn-Sachs, 1986] if,

$$x \in K,\; F(x) - F(y) \in -P \Longrightarrow F(\xi) - F(y) \in -P \;\forall\, \xi \in\,]x, y[.$$

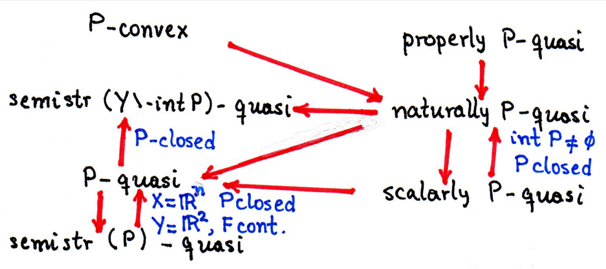
[R. Cambini, 1998] When $X = \mathbb{R}^n$, $Y = \mathbb{R}^2$, $P \subseteq \mathbb{R}^2$ polyhedral, $\mathrm{int}\, P \neq \emptyset$, $F : K \to \mathbb{R}^2$ continuous, both are equivalent.

One more Definition

7. semistrictly-($Y \setminus -\mathrm{int}\, P$)-quasiconvex at y [FB, 2004] if, $x \in K$,

$$F(x) - F(y) \notin \mathrm{int}\, P \Longrightarrow F(\xi) - F(y) \notin \mathrm{int}\, P \;\; \forall\, \xi \in \;]x, y[.$$

Teorema [FB, 2004]. Sean X, Y, K, P, F as above. We have:



P-quasiconv. implies semistrictly $(Y \setminus -\text{int } P)$-quasiconv.

Proof. Take any $x, y \in K$ such that $F(x) - F(y) \notin \text{int } P$, and suppose $\exists\, \xi \in\ ]x, y[$ satisfying $F(\xi) - F(y) \in \text{int } P$. If $F(x) - F(\xi) \in P$, the latter inclusion implies $F(x) - F(y) \in \text{int } P$ which cannot happen by the choice of x, y. Hence $F(x) - F(\xi) \notin P$. By a Lemma due to Bianchi-Hadjisavvas-Schaible (1997) ($a \not\geq 0$ $b < 0 \Rightarrow \exists c \not\geq 0$, $a \leq c$, $b \leq c$) there exists $c \notin P$ such that

$$F(x) - F(\xi) - c \in -P \quad \text{and} \quad F(y) - F(\xi) - c \in -P.$$

By the P-quasiconvexity of F, we conclude in particular $F(\xi) - F(\xi) - c = -c \in -P$ giving a contradiction. Consequently $F(\xi) - F(y) \notin \text{int } P$ for all $\xi \in\ ]x, y[$, proving the desired result.

Definition

Given $S \subseteq Y$, $K \subseteq X$ convex. The function $F : K \to Y$ is semistrictly (S)-quasiconvex at $y \in K$, if for every $x \in K$, $x \neq y$, $F(x) - F(y) \in -S \implies F(\xi) - F(y) \in -S \quad \forall\, \xi \in \,]x, y[$.
We say that F is semistrictly (S)-quasiconvex (on K) if it is at every $y \in K$.

$$F_1(x) = (e^{-x^2}, x^2), \quad x \in \mathbb{R}; \; F_2(x) = (\frac{1}{1+|x|^2}, |x|), \quad x \in \mathbb{R};$$

$$F_3(x_1, x_2) = \left(\frac{x_1^2}{1+x_1^2}, x_2^3\right), \quad (x_1, x_2) \in \mathbb{R}^2,$$

are semistrictly $(\mathbb{R}^2 \setminus -\mathrm{int}\, \mathbb{R}^2_+)$-quasiconvex.

Particular cases

$Y = \mathbb{R}$, $\mathbb{R}_+ \doteq [0, +\infty[$, $\mathbb{R}_{++} \doteq \ ]0, +\infty[$:
semistrict ($\mathbb{R}_+$)-quasiconvexity = quasiconvexity;
semistrict ($\mathbb{R}_{++}$)-quasiconvexity = semistrict quasiconvexity.

The previous definition is related to the problem of finding $\bar{x} \in X$ satisfying

$$\bar{x} \in K \text{ such that } F(x) - F(\bar{x}) \in S \ \forall\, x \in K.$$

The set of such $\bar{x}$ is denoted by E_S.

Set $\mathcal{L}_y \doteq \{x \in K : \ F(x) - F(y) \in -S\}$.

Proposition

Assume $0 \in S$ (for instance $S = Y \setminus -\mathrm{int}\, P$, $S = Y \setminus -P \setminus l(P)$); K convex; $F : K \to Y$, $y \in K$. The FAE:

- (a) F is semistrictly (S)-quasiconvex at y;
- (b) $\mathcal{L}_y$ is starshaped at y.

If $X = \mathbb{R}$, (b) may be substituted by the convexity of $\mathcal{L}_y$.

Proposition

Let S, K as above, and $\bar{x} \in K$ be a local S-minimal for F on K. Then, $\bar{x} \in E_S \iff F$ is semistrictly $(Y \setminus -S)$-quasiconvex at $\bar{x}$.

To fix ideas let $X = \mathbb{R}^n$, the asymptotic cone of C is

$$C^\infty \doteq \{v \in X : \exists\, t_n \downarrow 0, \exists\, x_n \in C,\ t_n x_n \to v\},$$

When C is closed and starshaped at $x_0 \in C$, one has

$$C^\infty = \bigcap_{t>0} t(C - x_0).$$

If C is convex the above expression is independent of $x_0 \in C$.

$$E_S \doteq \bigcap_{y \in K} \Big\{x \in K : F(x) - F(y) \in -S\Big\},$$

$$(E_S)^\infty \subseteq \bigcap_{y \in K} \Big\{x \in K : F(x) - F(y) \in -S\Big\}^\infty$$

$$(E_S)^\infty \subseteq \bigcap_{y \in K} \Big\{v \in K^\infty : F(y+\lambda v) - F(y) \in -S\ \forall\, \lambda > 0\Big\} \doteq R_S.$$

We introduce the following cones in order to deal with the case when K unbounded. Here $S \subseteq Y$,

$$R_P \doteq \bigcap_{y \in K} \Big\{ v \in K^\infty : \ F(y + \lambda v) - F(y) \in -P \ \ \forall\, \lambda > 0 \Big\},$$

$$R_S \doteq \bigcap_{y \in K} \Big\{ v \in K^\infty : \ F(y + \lambda v) - F(y) \in -S \ \ \forall\, \lambda > 0 \Big\}.$$

We recall that E_S denotes the set of $\bar{x} \in X$ satisfying

$$\bar{x} \in K \text{ such that } F(x) - F(\bar{x}) \in S \ \ \forall\, x \in K.$$

$$E_S \neq \emptyset \implies 0 \in S.$$

$$\mathcal{L}_y \doteq \{x \in K; F(x) - F(y) \in -S\}.$$

Theorem

K closed convex; P convex cone; $S \subseteq Y$ such that $S + P \subseteq S$; $F : K \to Y$ semistrictly (S)-quasiconvex and $\mathcal{L}_y$ is closed $\forall\, y \in K$. The following hold:

- $E_S + R_P = E_S$, $R_P \subseteq (E_S)^\infty \subseteq R_S$;
- if $E_S \neq \emptyset$ and either $X = \mathbb{R}$ or $Y = \mathbb{R}$ (with $P = [0, +\infty[$), $\Longrightarrow E_S$ is convex and $(E_S)^\infty = R_S$;
- $E_P \neq \emptyset \Longrightarrow (E_S)^\infty = R_S$, $(E_P)^\infty = R_P$.

Models: $S = P$, $S = Y \setminus -(P \setminus l(P))$, $S = Y \setminus -\mathrm{int}P$.

Proposition [FB-Vera, 2006]

$K \subseteq \mathbb{R}^n$ closed convex; $S \subseteq Y$; $F : K \to Y$ semistrictly (S)-quasiconvex and $\mathcal{L}_y$ closed for all $y \in K$.

- If $R_S = \{0\} \Longrightarrow (K_r \doteq K \cap \bar{B}(0,r))$

$$\exists\, r > 0, \forall\, x \in K \setminus K_r, \exists\, y \in K_r : F(y) - F(x) \notin S; \qquad (*)$$

- if $X = \mathbb{R}$ then (without the closedness of $\mathcal{L}_y$),

$$R_S = \{0\} \iff (*) \text{ holds.}$$

when $S = Y \setminus -\mathrm{int}\, P$, we denote $E_S = E_W$, $R_S = \tilde{R}_W$.

Theorem

$K \subseteq \mathbb{R}^n$ closed convex; $P \subseteq Y$ closed cone; $F : K \to Y$ semistrictly $(Y \setminus -\mathrm{int}\, P)$-quasiconvex with $G(y) = \{x \in K : F(x) - F(y) \notin \mathrm{int}\, P\}$ closed $\forall\, y \in K$. Then

$$\tilde{R}_W = \{0\} \Longrightarrow E_W \neq \emptyset \text{ and compact.}$$

Remarks

- Unfortunately, we do not know whether the condition $\tilde{R}_W = \{0\}$ is also necessary for the nonemptines and compactness of E_W in this general setting.
- convex case If $P = \mathbb{R}^m_+$ and each component of F is convex and lsc, the equivalence holds [Deng, 1998]. It will be extended for general cones latter on.
- a nonconvex case If $n = 1$ or $Y = \mathbb{R}$ ...

We set $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$,

Hypothesis on the cone P

$P \subseteq \mathbb{R}^m$ is a closed convex cone, $\mathrm{int}\, P \neq \emptyset$ (thus $P^* = \bigcup_{t>0} tB$ for some compact convex set B). We require that the set B_0 of extreme points of B is closed.

Obviously the polhyedral and the ice-cream cones satisfy the previous hypothesis.

Set ($S = \mathbb{R}^m \setminus -\mathrm{int}\ P$)

$$E_W \doteq \bigcap_{y\in K} \bigcup_{q\in B_0} \Big\{x \in K : \langle q, F(x) - F(y)\rangle \leq 0\Big\},$$

$$R_P = \bigcap_{y\in K} \bigcap_{\lambda>0} \bigcap_{q\in B_0} \Big\{v \in K^\infty : \langle q, F(y+\lambda v) - F(y)\rangle \leq 0\Big\},$$

Additionally, we also consider the cone

$$\tilde{R}_W = \bigcap_{y\in K} \bigcup_{q\in B_0} \Big\{v \in K^\infty : \langle q, F(y+\lambda v) - F(y)\rangle \leq 0 \ \forall\ \lambda > 0\Big\}.$$

Corollary: $h_q(x) = \langle q, F(x) \rangle, \; q \in P^*, x \in K$.

Assume h_q is convex for all $q \in B_0$; $F : K \to \mathbb{R}^m$ P-lsc. Then,

- if $E_W \neq \emptyset$,

$$\bigcap_{q \in B_0} \left\{ v \in K^\infty : \; h_q^\infty(v) \leq 0 \right\} \subseteq (E_W)^\infty \subseteq \bigcup_{q \in B_0} \left\{ v \in K^\infty : \; h_q^\infty(v) \leq 0 \right\};$$

- if $\mathrm{argmin}_K \; h_q \neq \emptyset$ for all $q \in B_0$,

$$(E_W)^\infty = \bigcup_{q \in B_0} \left\{ v \in K^\infty : \; h_q^\infty(v) \leq 0 \right\} = \tilde{R}_W.$$

Examples showing optimality of the assumptions

Example 3.1.

Take $P=\mathbb{R}^2_+$, $K=\mathbb{R}^2$, $f_1(x_1,x_2)=x_1^2$, $f_2(x_1,x_2)=e^{x_2}$. Then $f_1^\infty(v_1,v_2)=0$ if $v_1=0$, $f_1^\infty(v_1,v_2)=+\infty$ elsewhere; $f_2^\infty(v_1,v_2)=0$ if $v_2\leq 0$, $f_2^\infty(v_1,v_2)=+\infty$ elsewhere. Thus,

$$R_P=\{0\}\times\ ]-\infty,0],\ \tilde{R}_W=\Big(\{0\}\times\mathbb{R}\Big)\cup\Big(\mathbb{R}\times\ ]-\infty,0]\Big),$$

while $E_W=\{0\}\times\mathbb{R}=(E_W)^\infty$. Notice that $\operatorname{argmin}_K f_2=\emptyset$.

The convex case

Theorem [Deng, 1998; FB-Vera, 2006]

$K \subseteq \mathbb{R}^n$ closed convex; P closed convex cone as above. Assume $F : K \to \mathbb{R}^m$ is P-lsc such that $\langle q, F(\cdot)\rangle : K \to \mathbb{R}$ is convex $\forall \; q \in B_0$. The FAE:

(*a*) E_W is nonempty and compact;

(*b*) $\mathrm{argmin}_K \langle q, F(\cdot)\rangle$ is nonempty and compact for all $q \in B_0$;

(*c*) $\tilde{R}_w = \{0\}$;

The nonconvex case: non quasiconvexity

Theorem [FB-Vera, 2006]

$K \subseteq \mathbb{R}$ closed convex; $P \subseteq Y$ convex cone, $\mathrm{int}\, P \neq \emptyset$;
$F : K \to Y$ is semistrictly $(Y \setminus -\mathrm{int}\, P)$-quasiconvex such that $\mathcal{L}_y$ is closed $\forall\, y \in K$. Then, E_W is closed convex, and the FAE:

(a) $\tilde{R}_W = \{0\}$;

(b) $\exists\, r > 0,\ \forall\, x \in K \setminus K_r,\ \exists\, y \in K_r :\ F(y) - F(x) \in -\mathrm{int}\, P$, where $K_r = [-r, r] \cap K$;

(c) $E_W \neq \emptyset$ and bounded (it is already closed and convex).

When $P = \mathbb{R}^m_+$ some of the components of F may be not quasiconvex.

The nonconvex case: quasiconvexity

Theorem [FB, 2004; FB-Vera, 2006]

$Y = \mathbb{R}^n$, $K \subseteq \mathbb{R}$ is closed convex; $P \subseteq \mathbb{R}^m$ closed convex cone as above. Assume $\langle q, F(\cdot)\rangle : K \to \mathbb{R}$ is lsc and semistrictly quasiconvex $\forall\, q \in B_0$. The FAE:

(a) E_W is a nonempty compact convex set;

(b) $\mathrm{argmin}_K \langle q, F(\cdot)\rangle$ is a nonempty compact convex set for all $q \in B_0$;

(c) $\exists\, r > 0$, $\forall\, x \in K \setminus K_r$, $\exists\, y \in K_r$ ($K_r = [-r, r] \cap K$):

$$\langle q, F(y) - F(x)\rangle < 0 \;\; \forall\, q \in B_0.$$

Examples showing optimality of the assumptions

Example 4.1.

Consider $P = \mathbb{R}^2_+$, $K = \mathbb{R}$, $F(x) = (\sqrt{|x|}, \frac{x}{1+|x|})$, $x \in \mathbb{R}$. Here, $E_W = \,]-\infty, 0]$.

Example 4.2.

Consider $P = \mathbb{R}^2_+$, $F = (f_1, f_2)$, $K = [0, +\infty[$ where,

$$f_1(x) = \begin{cases} 2, & \text{if } x \notin [1,2] \\ 1, & \text{if } x \in [1,2] \end{cases} \qquad f_2(x) = \begin{cases} -e^{-x+5}, & \text{if } x \geq 5 \\ 4 - x, & \text{if } x < 5 \end{cases}$$

Here, $E_W = [1, +\infty[$.

Conjecture:

Assume that each $f_i : K \subseteq \mathbb{R}^n \to \mathbb{R}$ is semistrictly quasiconvex and lsc, $i = 1, \ldots, m$. The FAE:

- E_W is nonempty and compact;
- each $\mathrm{argmin}_K f_i$ is nonempty and compact.

The starting point: linear case

Theorem [Gordan Paul, 1873] Let A matrix.

Then, exactly one of the following sistems has solution:

(*I*) $Ax < 0$;

(*II*) $A^\top p = 0$, $p \geq 0$, $p \neq 0$.

The convex case

Theorem [Fan-Glicksberg-Hoffman, 1957] Let $K \subseteq \mathbb{R}^n$ convex, $f_i : K \to \mathbb{R}$, $i = 1, \ldots, m$, convex. Then, exactly one of the following two sistems has solution:

(I) $f_i(x) < 0$, $i = 1, \ldots, m$, $x \in K$;

(II) $p \in \mathbb{R}^m_+ \setminus \{0\}$, $\sum_{i=1}^m p_i f_i(x) \geq 0 \; \forall \, x \in K$.

Sketch of Proof. Set $F = (f_1, \ldots, f_m)$.

$$\text{Not } (I) \iff F(K) \cap (-\text{int}\, \mathbb{R}^m_+) = \emptyset \iff (F(K) + \mathbb{R}^m_+) \cap (-\text{int}\, \mathbb{R}^m_+) = \emptyset$$

$$\Updownarrow \quad (F(K) + \mathbb{R}^m_+ \text{ is convex} \implies (II))$$

$$\overline{\text{cone}}(F(K) + \mathbb{R}^m_+) \cap (-\text{int}\, \mathbb{R}^m_+) = \emptyset$$

Let P closed convex cone with $\text{int } P \neq \emptyset$

$$F(K) \approx A \subseteq Y, \;\; \mathbb{R}^n_+ \approx P$$

(I) $A \cap (-\text{int } P) \neq \emptyset$,

(II) $\text{co}(A) \cap (-\text{int } P) = \emptyset$.

Trivial part (I) y (II) $\Longrightarrow$ absurd.

Non trivial part: Hipothesis (¿ ?)

$$A \cap (-\text{int } P) = \emptyset \Longrightarrow \text{co}(A) \cap (-\text{int } P) = \emptyset.$$

$$A \cap (-\text{int } P) = \emptyset \Longleftrightarrow \overline{\text{cone}}(A + P) \cap (-\text{int } P) = \emptyset.$$

It suffices the convexity of $\overline{\text{cone}}(A + P)$!!

Definition: Let $P \subseteq Y$ closed convex cone, $\mathrm{int}\, P \neq \emptyset$.

The set $A \subseteq Y$ is:

(*a*) generalized subconvexlike [Yang-Yang-Chen, 2000] if $\exists\, u \in \mathrm{int}\, P,\ \forall\, x_1,\ x_2 \in A,\ \forall\, \alpha \in\ ]0,1[,\ \forall\, \varepsilon > 0,\ \exists\, \rho > 0$ such that

$$\varepsilon u + \alpha x_1 + (1-\alpha)x_2 \in \rho A + P; \qquad (1)$$

(*b*) presubconvexlike [Zeng, 2002] if $\exists\, u \in Y,\ \forall\, x_1,\ x_2 \in A,\ \forall\, \alpha \in\ ]0,1[,\ \forall\, \varepsilon > 0,\ \exists\, \rho > 0$ such that (1) holdse;

(*c*) nearly subconvexlike [Sach, 2003; Yang-Li-Wang, 2001] if $\overline{\mathrm{cone}}(A+P)$ is convex.

(*a*), (*b*), (*c*) are equiv. [FB-Hadjisavvas-Vera, 2007].

$$\text{cone}_+(A + \text{int } P) \text{ is convex } \iff \text{cone}(A + \text{int } P) \text{ is convex.}$$

$$\Longrightarrow \overline{\text{cone}}(A + \text{int } P) = \overline{\text{cone}}(\overline{A + \text{int } P}) = \overline{\text{cone}}(\overline{A + \overline{\text{int } P}}) =$$

$$\overline{\text{cone}}(A + P) = \overline{\text{cone}_+}(A + P) \text{ is convex.}$$

Also,

$$\text{int}(\overline{\text{cone}_+(A + P)}) = \text{int}(\overline{\text{cone}_+(A) + P}) = \text{cone}_+(A) + \text{int } P =$$

$$\text{cone}_+(A + \text{int } P) \text{ is convex. Consequently,}$$

$$\overline{\text{cone}}(A + P) \text{ is convex } \iff \text{cone}(A + \text{int } P) \text{ is convex.}$$

Here, $\overline{\text{cone}(M)} = \overline{\text{cone}_+(M)}$.

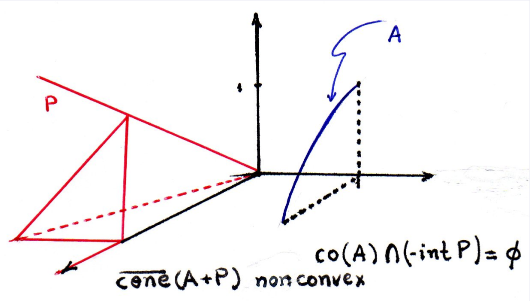
$$\text{cone}(M) = \bigcup_{t \geq 0} tM, \ \text{cone}_+(M) \doteq \bigcup_{t > 0} tM, \overline{\text{cone}}(M) = \overline{\text{cone}(M)}.$$

Theorem [Yang-Yang-Chen, 2000; Yang-Li-Wang, 2001]

$P \subseteq Y$ as above, $A \subseteq Y$. Assume $\overline{\text{cone}}(A + P)$is convex. Then

$$A \cap (-\text{int } P) = \emptyset \Longrightarrow \text{co}(A) \cap (-\text{int } P) = \emptyset.$$

Example: [FB-Hadjisavvas-Vera, 2007]



Def: A cone $K \subseteq Y$ is called "pointed" if

$x_1 + \cdots + x_k = 0$ is impossible for $x_1, x_2, \ldots, x_k$ in K unless $x_1 = x_2 = \cdots = x_k = 0$. ($\Longleftrightarrow \operatorname{co} K \cap (-\operatorname{co} K) = \{0\}$).

Our first main result is the following:

Theorem [FB-Hadjisavvas-Vera, 2007]

: $\emptyset \neq A \subseteq Y$, $P \subseteq Y$ convex closed cone, $\operatorname{int} P \neq \emptyset$. The FAE:

(a) $\operatorname{cone}(A + \operatorname{int} P)$ is pointed;

(b) $\operatorname{co}(A) \cap (-\operatorname{int} P) = \emptyset$.

Sketch of proof

We first prove $\mathrm{cone}(A + \mathrm{int}\, P)$ is pointed $\implies A \cap (-\mathrm{int}\, P) = \emptyset$. If $\exists\, x \in A \cap (-\mathrm{int}\, P)$, then $x = 2(x - \frac{x}{2}) \in \mathrm{cone}(A + \mathrm{int}\, P)$ and $-x = x + (-2x) \in A + \mathrm{int}\, P \subseteq \mathrm{cone}(A + \mathrm{int}\, P)$. By pointedness, $0 = x + (-x)$ implies $x = 0 \in \mathrm{int}\, P$, a contradiction.
Now assume that (a) holds. If (b) does not hold, $\exists\, x \in -\mathrm{int}\, P$ such that $x = \sum_{i=1}^{m} \lambda_i a_i$ with $\sum_{i=1}^{m} \lambda_i = 1$, $\lambda_i > 0$, $a_i \in A$. Thus, $0 = \sum_{i=1}^{m} \lambda_i (a_i - x)$. Using (a), $\lambda_i (a_i - x) = 0\ \forall\, i = 1, \ldots, m$, a contradiction. Conversely, assume (b) holds. If $\mathrm{cone}(A + \mathrm{int}\, P)$ is not pointed, then $\exists\, x_i \in \mathrm{cone}(A + \mathrm{int}\, P) \setminus \{0\}$, $i = 1, 2, \ldots n$, $\sum_{i=1}^{n} x_i = 0$. So, $x_i = \lambda_i (y_i + u_i)$ with $\lambda_i > 0$, $y_i \in A$ and $u_i \in \mathrm{int}\, P$. Hence $\sum_{i=1}^{n} \lambda_i y_i = -\sum_{i=1}^{n} \lambda_i u_i$. Setting $\mu_i = \lambda_i / \sum_{j=1}^{n} \lambda_j$ we get $\sum_{i=1}^{n} \mu_i y_i = -\sum_{i=1}^{n} \mu_i u_i \in \mathrm{co}(A) \cap (-\mathrm{int}\, P)$, a contradiction.

The optimal 2D alternative theorem

Theorem [FB-Hadjisavvas-Vera, 2007]

Let $P \subseteq \mathbb{R}^2$ be a cone as before with $\mathrm{int}\, P \neq \emptyset$, and $A \subseteq \mathbb{R}^2$ be satisfying $A \cap (-\mathrm{int}\, P) = \emptyset$. The following hold:

$$\mathrm{co}(A) \cap (-\mathrm{int}\, P) = \emptyset \iff \mathrm{cone}(A + P) \text{ is convex } \iff$$

$$\mathrm{cone}(A + \mathrm{int}\, P) \text{ is convex } \iff \mathrm{cone}(A) + P \text{ is convex } \iff$$

$$\overline{\mathrm{cone}}(A + P) \text{ is convex.}$$

We are in $\mathbb{R}^2$, $\mathrm{int}(\mathrm{cone}_+(A + P)) \cup \{0\} = \mathrm{cone}(A + \mathrm{int}\, P) \subseteq$

$$\mathrm{cone}(A + \mathrm{int}\, P) \subseteq \mathrm{cone}(A + P) \subseteq \mathrm{cone}(A) + P \subseteq \overline{\mathrm{cone}}(A + P).$$

Remark

$$A\cap(-\text{int } P) = \emptyset \;\&\; \text{cone}(A+P) \text{ is convex} \Longleftrightarrow \text{co}(A)\cap(-\text{int } P) = \emptyset.$$

Theorem [FB-Hadjisavvas-Vera, 2007] Y LCTVS.

$P \subseteq Y$ closed convex cone, $\operatorname{int} P \neq \emptyset$ and $\operatorname{int} P^* \neq \emptyset$. The FAE:

(*a*) for every $A \subseteq Y$ one has

$$\mathrm{co}(A) \cap (-\operatorname{int} P) = \emptyset \Rightarrow \overline{\mathrm{cone}}(A + P) \ \text{ is convex;}$$

(*b*) for every $A \subseteq Y$ one has

$$\mathrm{co}(A) \cap (-\operatorname{int} P) = \emptyset \Rightarrow \mathrm{cone}\,(A) + P \ \text{ is convex;}$$

(*c*) for every $A \subseteq Y$ one has

$$\mathrm{co}(A) \cap (-\operatorname{int} P) = \emptyset \Rightarrow \mathrm{cone}\,(A + \operatorname{int} P) \ \text{ is convex;}$$

(*d*) Y is at most two-dimensional.

The assumption $\mathrm{int}\ P^* \neq \emptyset$ (which corresponds to pointedness of P when Y is finite-dimensional) cannot be removed.
Indeed, let $P = \{y \in Y : \langle p^*, y\rangle \geq 0\}$ where $p^* \in Y^* \backslash \{0\}$. Then $P^* = \mathrm{cone}\,(\{p^*\})$, $\mathrm{int}\ P^* = \emptyset$. For any nonempty $A \subseteq Y$, the set $\mathrm{cone}(A + \mathrm{int}\ P)$ is convex if $A \cap (-\mathrm{int}\ P) = \emptyset$
$(\iff A \subseteq P \iff \mathrm{co}(A) \cap (-\mathrm{int}\ P) = \emptyset)$.
Thus, the previous implication holds independently of the dimension of the space Y.

Characterization of weakly efficient solutions via linear scalarization

$K \subseteq \mathbb{R}^n$ convex and P as above. Given $F : K \to \mathbb{R}^m$, we consider

$$\bar{x} \in K : \; F(x) - F(\bar{x}) \notin -\mathrm{int}\, P, \;\; \forall\, x \in K,$$

Clearly, $\bar{x} \in E_W \iff (F(K) - F(\bar{x})) \cap -\mathrm{int}\, P = \emptyset.$

Teorema[FB-Hadjisavvas-Vera, 2007]: The FAE

(a)

$$\bar{x} \in \bigcup_{p^* \in P^*, p^* \neq 0} \mathrm{argmin}_K \langle p^*, F(\cdot) \rangle;$$

(b) $\mathrm{cone}(F(K) - F(\bar{x}) + \mathrm{int}\, P)$ is pointed.

Theorem [FB-Hadjisavvas-Vera, 2007]

Set $m = 2$. The FAE:

(a)

$$\bar{x} \in \bigcup_{p^* \in P^*, p^* \neq 0} \mathrm{argmin}_K \langle p^*, F(\cdot) \rangle;$$

(b) $\bar{x} \in E_W$ and $\mathrm{cone}(F(K) - F(\bar{x}) + \mathrm{int}\, P)$ is convex.

(c) $\bar{x} \in E_W$ and $\mathrm{cone}(F(K) - F(\bar{x}) + P)$ is convex.

(d) $\bar{x} \in E_W$ and $\mathrm{cone}(F(K) - F(\bar{x})) + P$ is convex.

$$\mathrm{cone}(A) = \bigcup_{t \geq 0} tA.$$

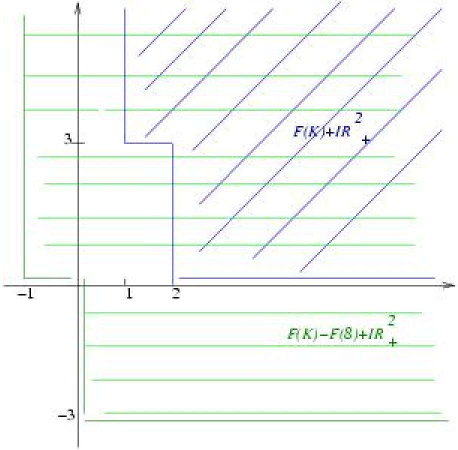
Example

Consider $F = (f_1, f_2)$, $K = [0, +\infty[$ where,

$$f_1(x) = \begin{cases} 2, & \text{if } x \notin [1,2] \\ 1, & \text{if } x \in [1,2] \end{cases} \qquad f_2(x) = |x - 5|.$$

Here, $E_W = [1, 8]$, whereas

$$\bigcup_{p^* \in \mathbb{R}^2_+, p^* \neq 0} \operatorname{argmin}_K \langle p^*, F(\cdot) \rangle = [1, 5].$$


F(K)+IR
2
+
3
−1
1
2
F(K)−F(8)+IR
2
+
−3

Open problem

to find an assumption convexity of ?? (*) such that

$$(*)\ \ \&\ \ A \cap (-\text{int } P) = \emptyset \implies \text{co}(A) \cap (-\text{int } P) = \emptyset.$$

At least for $A \approx G(K)$ some class of vector functions $G : K \to Y$.

Characterizing the Fritz-John type optimality conditions in VO

Take X normed space. It is known that if $\bar{x}$ is a local minimum point for (differentiable) $F : K \to \mathbb{R}$ on K, then

$$\nabla F(\bar{x}) \in (T(K;\bar{x}))^*.$$

Here, $T(C;\bar{x})$ denotes the contingent cone of C at $\bar{x} \in C$,

$$T(C;\bar{x}) = \Big\{ v \in X : \exists\, t_k \downarrow 0,\ v_k \in X, v_k \to v,\ \bar{x} + t_k v_k \in C\ \forall\, k \Big\}.$$

How to extend to the vector case ?

$K \subseteq X$ closed; $F : K \to \mathbb{R}^m$; $P \subseteq \mathbb{R}^m$, $\operatorname{int} P \neq \emptyset$, a vector $\bar{x} \in K$ is a local weakly efficient solution for F on K ($\bar{x} \in E_W^{loc}$), if there exists an open neighborhood V of $\bar{x}$ such that

$$(F(K \cap V) - F(\bar{x})) \cap (-\operatorname{int} P) = \emptyset.$$

We say that a function $h : X \to \mathbb{R}$ admits a Hadamard directional derivative at $\bar{x} \in X$ in the direction v if

$$\lim_{(t,u)\to(0^+,v)} \frac{h(\bar{x} + tu) - h(\bar{x})}{t} \in \mathbb{R}.$$

In this case, we denote such a limit by $dh(\bar{x}; v)$.

If $F = (f_1, \ldots, f_m)$, we set

$$\mathcal{F}(v) \doteq ((df_1(\bar{x}; v), \ldots, df_m(\bar{x}; v)),$$

$$\mathcal{F}(T(K; \bar{x})) = \{\mathcal{F}(v) \in \mathbb{R}^m : \; v \in T(K; \bar{x})\}.$$

It is known that if $df_i(\bar{x}; \cdot)$, $i = 1, \ldots, m$ do exist in $T(K; \bar{x})$, and $\bar{x} \in E_W^{loc}$, then

$$(df_1(\bar{x}; v), \ldots, df_m(\bar{x}; v)) \in \mathbb{R}^m \setminus -\mathrm{int}\, P, \;\; \forall\, v \in T(K; \bar{x}),$$

or equivalently, $\mathcal{F}(T(K; \bar{x})) \cap (-\mathrm{int}\, P) = \emptyset.$

Theorem [FB-Hadjisavvas-Vera, 2007][$Y = \mathbb{R}^m$]

Under the assumptions above, the FAE:

- $\exists(\alpha_1^*, \ldots, \alpha_m^*) \in P^* \setminus \{0\},\ \alpha_1^* df_1(\bar{x}, v) + \cdots + \alpha_m^* df_m(\bar{x}, v) \geq 0\ \ \forall\ v \in T(K; \bar{x})$;
- $\mathrm{cone}(\mathcal{F}(T(K; \bar{x})) + \mathrm{int}\ P)$ is pointed.

A more precise formulation may be obtained when $m = 2$.

Theorem [FB-Hadjisavvas-Vera, 2007][$m = 2$]

The FAE:

- $\exists\ (\alpha_1^*, \alpha_2^*) \in P^* \setminus \{0\},\ \ \alpha_1^* df_1(\bar{x}, v) + \alpha_2^* df_2(\bar{x}, v) \geq 0\ \ \forall\ v \in T(K; \bar{x})$;
- $\mathcal{F}(T(K; \bar{x})) \cap (-\mathrm{int}\ P) = \emptyset$ & $\mathrm{cone}(\mathcal{F}(T(K; \bar{x})) + \mathrm{int}\ P)$ is convex.

$P = \mathbb{R}^m_+$, $f_i : \mathbb{R}^n \to \mathbb{R}$ is diff. for $i = 1, \ldots, m$. Then $df_i(\bar{x}, v) = \langle \nabla f_i(\bar{x}), v \rangle$,

$$\mathcal{F}(v) = (\langle \nabla f_1(\bar{x}), v \rangle, \ldots, \langle \nabla f_m(\bar{x}), v \rangle).$$

Moreover,

$$\exists \alpha^* \in \mathbb{R}^m_+ \setminus \{0\},\ \alpha^*_1 df_1(\bar{x}, v) + \cdots + \alpha^*_m df_m(\bar{x}, v) \geq 0\ \forall\, v \in T(K; \bar{x})$$

$$\Updownarrow$$

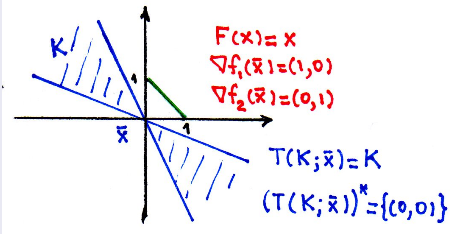
$$\mathrm{co}(\{\nabla f_i(\bar{x}) :\ i = 1, \ldots, m\}) \cap (T(K; \bar{x}))^* \neq \emptyset$$

This is not always a necessary optimality condition.
In fact !!

Example [FB-Hadjisavvas-Vera, 2007]

$K = \{(x_1, x_2) : (x_1 + 2x_2)(2x_1 + x_2) \leq 0\}$. Take $f_i(x_1, x_2) = x_i$, $\bar{x} = (0, 0) \in E_W$: $T(K; \bar{x}) = K$ is nonconvex; $\mathcal{F}(v) = v$; $(T(K; \bar{x}))^* = \{(0, 0)\}$, and

$$\mathrm{co}(\{\nabla f_i(\bar{x}) : \ i = 1, 2\}) \cap (T(K; \bar{x}))^* = \emptyset.$$



In the example,

$$\mathrm{cone}(\mathcal{F}(T(K;\bar{x})) + \mathbb{R}^2_+) = \bigcup_{t\geq 0} t(T(K;\bar{x}) + \mathbb{R}^2_+) \text{ is nonconvex.}$$

On the other hand, due to the linearity of $\mathcal{F}$ (when each f_i is differentiable), if $T(K;\bar{x})$ is convex then

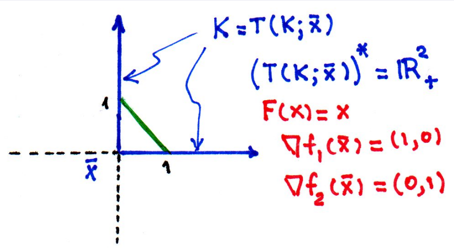
$$\mathrm{cone}(\mathcal{F}(T(K;\bar{x})) + \mathbb{R}^m_+) = \bigcup_{t\geq 0} t(\mathcal{F}(T(K;\bar{x})) + \mathbb{R}^m_+) \text{ is also convex.}$$

This fact was point out earlier in [Wang, 1988], i.e., if $T(K;\bar{x})$ is convex the condition above is a necessary optimality condition.

The convexity of $T(K;\bar{x})$ is the only case ??

NO !

Example [FB-Hadjisavvas-Vera, 2007]



Thus,

$$\mathrm{co}(\{\nabla f_i(\bar{x}) : \ i = 1,2\}) \cap (T(K;\bar{x}))^* \neq \emptyset. \ \textit{And}$$

$$\bigcup_{t \geq 0} t(\mathcal{F}(T(K;\bar{x})) + \mathbb{R}^2_+) = \bigcup_{t \geq 0} t(T(K;\bar{x}) + \mathbb{R}^2_+) \ \text{is convex}.$$

A non linear scalarization procedure

Def.: Let $a \in Y$, $e \in \operatorname{int} P$.

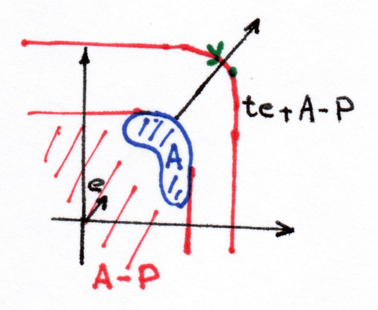
Define $\xi_{e,a} : Y \longrightarrow \mathbb{R} \cup \{-\infty\}$, by

$$\xi_{e,a}(y) = \inf\{t \in \mathbb{R} : y \in te + a - P\}.$$

Def. $A \subseteq Y$, $\xi_{e,A} : Y \to \mathbb{R} \cup \{-\infty\}$:

$$\xi_{e,A}(y) = \inf\{t \in \mathbb{R} \colon y \in te + A - P\}.$$

$$\xi_{e,A}(y) = \inf_{a \in A} \xi_{e,a}(y). \tag{1}$$



Lemma [Hernández-Rodriguez, 2007]: Let $\emptyset \neq A \subseteq Y$ and P as above.

Then, $A - P \neq Y \iff \xi_{e,A}(y) > -\infty \ \forall\, y \in Y$.

By taking into account that

$$\mathrm{int}(\overline{A - P}) = \mathrm{int}(A - P) = A - \mathrm{int}P, \quad \overline{A - P} = \overline{A - \mathrm{int}P},$$

one can prove,

Lemma: Let $A \subseteq Y$, $r \in \mathbb{R}$, $y \in Y$.

Then

(a) $\xi_{e,A}(y) < r \Leftrightarrow y \in re + A - \mathrm{int}(P)$;

(b) $\xi_{e,A}(y) \leq r \Leftrightarrow y \in re + \overline{A - P}$;

(c) $\xi_{e,A}(y) = r \Leftrightarrow y \in re + \partial(A - P)$.

Corollary: Let $\emptyset \neq P \subseteq Y$ closed convex proper cone.

(a) If $\mathrm{int}\, P \neq \emptyset$ and $E_W \neq \emptyset$, then

$$E_W = E(\xi_{e,f(E_W)} \circ f, K) = \bigcup_{x \in E_W} E(\xi_{e,f(x)} \circ f, K).$$

If in addition $E(\xi_{e,f(x)} \circ f, K) \neq \emptyset$ for some $x \in K$, then

$$E_W = \bigcup_{x \in K} E(\xi_{e,f(x)} \circ f, K);$$

(b) if $E \neq \emptyset$, then

$$E = \bigcup_{x \in E} E(\xi_{e,f(x)} \circ f, K) \subseteq E(\xi_{e,f(E)} \circ f, K);$$

The positive orthant

Example 1.

Consider $F(x) = (x, \sqrt{1+x^2})$, $x \in K = \mathbb{R}$. Here, $E_W = \,]-\infty, 0]$. However, if $p_1^* > p_2^* > 0$, then

$$\inf_{x \in \mathbb{R}} \langle p^*, F(x) \rangle = -\infty, \; p^* = (p_1^*, p_2^*).$$

Example 2.

Consider $F = (f_1, f_2)$, $K = [0, +\infty[$ where,

$$f_1(x) = \begin{cases} 2, & \text{if } x \notin [1,2] \\ 1, & \text{if } x \in [1,2] \end{cases} \quad f_2(x) = |x - 5|. \text{ Here } E_W = [1, 8].$$

Theorem [FB-Vera, 2008]

$K \subseteq \mathbb{R}$ is closed convex; $f_i : K \to \mathbb{R}$ is lsc and quasiconvex for all $i = 1, \ldots, m$. The following assertions hold:

(*a*) if $\emptyset \neq E_W \neq \mathbb{R}$, then there exists j such that $\mathrm{argmin}_K f_j \neq \emptyset$;

(*b*) if $K \neq \mathbb{R}$: then $E_W \neq \emptyset \iff \exists\, j,\ \mathrm{argmin}_K f_j \neq \emptyset$.

Theorem [FB-Vera, 2008]

$K \subseteq \mathbb{R}$ is closed convex; $f_i : K \to \mathbb{R}$ is lsc and semistrictly quasiconvex for all $i = 1, \ldots, m$. Assume $E_W \neq \emptyset$. Then, either

$$E_W = \mathbb{R} \ \text{ or } \ E_W = \mathrm{co}\Big(\bigcup_{j \in J} \mathrm{argmin}_K f_j\Big) + R_W.$$

The bicriteria case

We consider $F : K \subseteq \mathbb{R} \to \mathbb{R}^2$ such that

$$[\alpha_1, \beta_1] \doteq \mathrm{argmin}_K f_1, \ [\alpha_2, \beta_2] \doteq \mathrm{argmin}_K f_2,$$

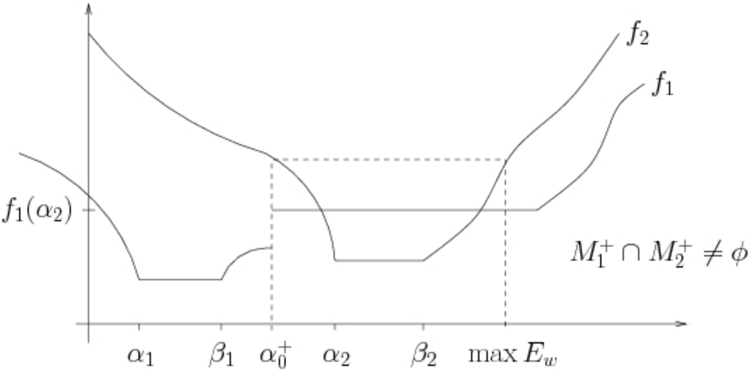
$$-\infty < \alpha_1 \leq \beta_1 < \alpha_2 \leq \beta_2 < +\infty.$$

Set

$$A_+ \doteq \{x \in [\beta_1, \alpha_2] : \ f_1(x) = f_1(\alpha_2)\},$$

$$A_- \doteq \{x \in [\beta_1, \alpha_2] : \ f_2(x) = f_2(\beta_1)\}.$$

$$\gamma_+ = \begin{cases} f_2(\alpha_0^+), & A_+ = \ ]\alpha_0^+, \alpha_2] \\ \lambda_+, & A_+ = \ [\alpha_0^+, \alpha_2] \end{cases} \qquad \lambda_+ \doteq \lim_{t \downarrow 0} f_2(\alpha_0^+ - t).$$



$$M_1^+ \doteq \{x \in K : \; x > \beta_2, \; f_1(x) = f_1(\alpha_2)\},$$
$$M_2^+ \doteq \{x \in K : \; x > \beta_2, f_2(x) = \gamma_+\}.$$

Theorem [FB-Vera, 2008]

$K \subseteq \mathbb{R}$ is closed convex; $f_i : K \to \mathbb{R}$ is lsc and quasiconvex for all $i = 1, 2$. Then A_+ and A_- are convex and nonempty. Moreover, we also have:

(a) $\bar{x} > \beta_2$: if $A_+ = \,]\alpha_0^+, \alpha_2]$, $\alpha_0^+ \geq \beta_1$, then

$$\bar{x} \in E_W \Longleftrightarrow f_2(\bar{x}) \leq f_2(\alpha_0^+), f_1(\bar{x}) = f_1(\alpha_2);$$

(b) $\bar{x} > \beta_2$: if $A_+ = [\alpha_0^+, \alpha_2]$, $\alpha_0^+ > \beta_1$, then

$$\bar{x} \in E_W \Longleftrightarrow f_2(\bar{x}) \leq \lambda_+, f_1(\bar{x}) = f_1(\alpha_2);$$

where $\lambda_+ \doteq \lim_{t \downarrow 0} f_2(\alpha_0^+ - t) = \inf_{y < \alpha_0^+} f_2(y)$.

Theorem [continued...]

(c) $\bar{x} < \alpha_1$: if $A_- = [\beta_1, \alpha_0^-[$, $\alpha_0^- \leq \alpha_2$, then

$$\bar{x} \in E_W \iff f_1(\bar{x}) \leq f_1(\alpha_0^-),\ f_2(\bar{x}) = f_2(\beta_1)\}$$

(d) $\bar{x} < \alpha_1$: if $A_- = [\beta_1, \alpha_0^-]$, $\alpha_0^- < \alpha_2$, then

$$\bar{x} \in E_W \iff f_1(\bar{x}) \leq f_1(\lambda_-),\ f_2(\bar{x}) = f_2(\beta_1)\}$$

where $\lambda_- \doteq \lim_{t\downarrow 0} f_2(\alpha_0^- + t)$.

Theorem [FB-Vera, 2008]

$K \subseteq \mathbb{R}$ convex closed; $f_i : K \to \mathbb{R}$ be lsc and quasiconvex for $i = 1, 2$.

(*a*) If f_2 is semistrictly quasiconvex and $M_1^+ \cap M_2^+ \neq \emptyset$, then $E_w = [\alpha_1, \bar{x}]$, where $\bar{x} \in K$ solves the system

$$\bar{x} > \beta_2 \;\; f_1(\bar{x}) = f_1(\alpha_2), \;\; f_2(\bar{x}) = \gamma_+.$$

(*b*) If f_1 is semistrictly quasiconvex and $M_1^- \cap M_2^- \neq \emptyset$, then $E_w = [\bar{x}, \beta_2]$, here $\bar{x} \in K$ solves the system

$$\bar{x} < \alpha_1 \;\; f_2(\bar{x}) = f_2(\beta_1), \;\; f_1(\bar{x}) = \gamma_-.$$

Example.

Consider $F = (f_1, f_2)$, $K = [0, +\infty[$,

$$f_1(x) = \begin{cases} 2, & \text{if } x \notin [1,2] \\ 1, & \text{if } x \in [1,2] \end{cases} \qquad f_2(x) = |x - 5|.$$

Here, $E = \{2, 5\}$, $E_W = [1, 8]$.

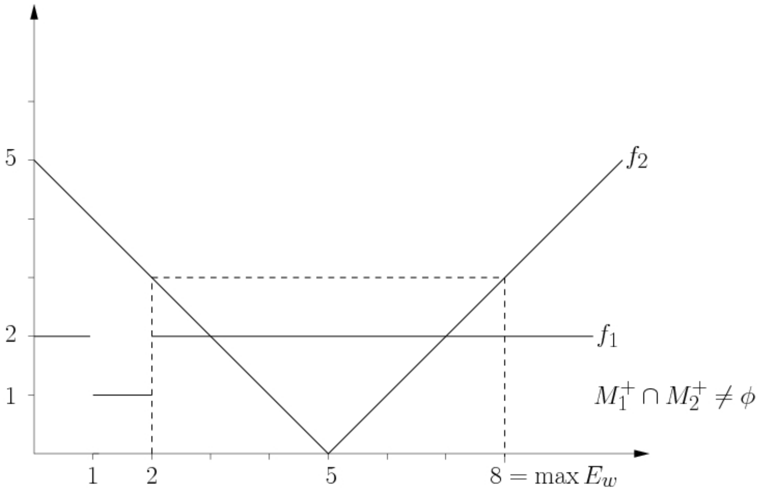

5
2
1
f_2
f_1
$M_1^+ \cap M_2^+ \neq \phi$
1
2
5
$8 = \max E_w$

TABLE 1

error	total cpu time	γ_+/iterations	$\max E_w$/iterations
10^{-3}	0.0150000000	2.9992675781/12	7.9993314775/40
10^{-4}	0.0160000000	2.9999084430/15	7.9999226491/42
10^{-5}	0.0160000000	2.9999942780/19	7.9999965455/45
10^{-6}	0.0160000000	2.9999992847/22	7.9999993877/49

Example.

Let $K=[0,+\infty[$,

$$f_1(x)=\begin{cases} 2 & \text{si } x<1,\\ 1 & \text{if} \quad x\in[1,2],\\ 2 & \text{if} \quad x\in]2,7[,\\ \sqrt{x-7}+2 & \text{if} \quad x>7, \end{cases}$$

$$f_2(x)=\begin{cases} 6-x & \text{if} \quad x<4,\\ e^{-(x-4)^2}+3 & \text{if} \quad x\geq 4, \end{cases}$$

Here $E_w=[0,7]$.

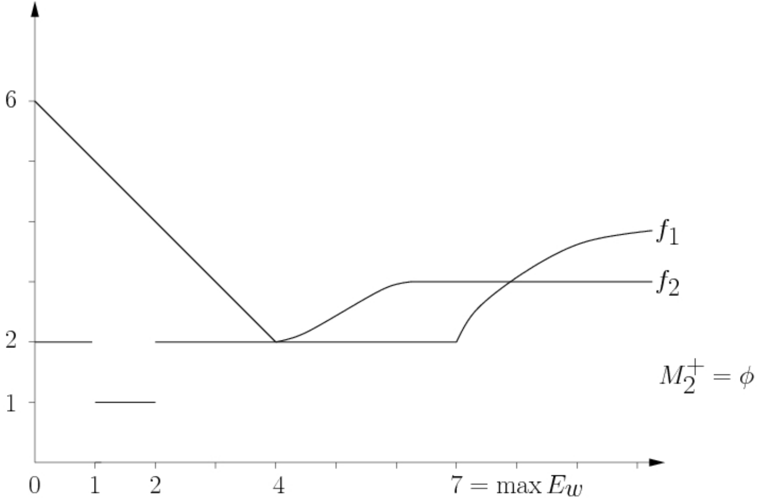

6
2
1
0
1
2
4
7 = max E_w
f_1
f_2
$M_2^+ = \phi$

TABLE2

error	total cpu time	γ_+/iterations	$\max E_w$/iterations
10^{-3}	0.0140000000	3.9990234375/11	6.9999681538/40
10^{-4}	0.0160000000	3.9993389648/15	6.9999908912/42
10^{-5}	0.0160000000	3.9999923706/18	6.9999997729/48
10^{-6}	0.0160000000	3.9999990463/21	6.9999997729/48

Multicriteria case

We describe E_W in the multicriteria case, that is when $m > 2$, since

$$E_W = \bigcup \{E_W(I) : \; I \subseteq \{1, 2 \ldots, m\}, \; |I| \leq 2\}, \qquad (1)$$

where $E_W(I)$ is the set of $\bar{x}$ solutions to the subproblem

$$\bar{x} \in K : \; F_I(x) - F_I(\bar{x}) \notin -\mathrm{int}\, \mathbb{R}^{|I|}_{+} \;\; \forall\, x \in K.$$

Here, $F_I = (f_i)_{i \in I}$ and $\mathbb{R}^{|I|}_{+}$ is the positive orthant in $\mathbb{R}^{|I|}$. One inclusion in (1) trivially holds since $E_W(I) \subseteq E_W(I')$ if $I \subseteq I'$; the other is a consequence of the following Helly's theorem since each f_i is quasiconvex.

Helly's theorem

Let C_i, $i = 1, \ldots, m$, be a collection of convex sets in $\mathbb{R}^n$. If every subcollection of $n + 1$ or fewer of these C_i has a nonempty intersection, then the entire collection of the m sets has a nonempty intersection.

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