Generalized Convexity in Nonlinear Elasticity with an Application to Unilateral Contact

Joachim Gwinner & Karl Dvorsky
Universität der Bundeswehr München
Department of Aerospace Engineering
D-85577 Neubiberg/München, Germany
joachim.gwinner@unibw-muenchen.de

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Dedicated to the memory of our colleague
Professor Dr. Tamas Rapcsák

Outline

I Short review of direct methods of convex analysis in calculus of variations

II Some generalized convexity concepts in nonlinear elasticity

III Minimization of non(quasi)convex integrands: Young measures and $\mathcal{A}$-convex relaxations

IV Application to unilateral contact: existence of minimizer, a penalty method, Euler-Lagrange equation

Classical direct methods in Calc Var.

Recall the classical direct methods of convex analysis

Consider functional

$$I(u) = \int_\Omega f(x, u(x), \nabla u(x)) \, \mathrm{d}x$$

Here $\Omega \subset \mathbb{R}^d$ bounded Lipschitz domain,

$f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \longrightarrow \mathbb{R}$ Lebesgue measurable

$u : \Omega \longrightarrow \mathbb{R}^m$ belongs to
a reflexive Banach space $X \subset W^{1,1}(\Omega, \mathbb{R}^m)$.

Variational problem: Find minimizer of $I(u)$
in some class of feasible functions contained in X

Interpretation in elasticity

Here $d = m$

u stands for the displacement field of an elastic body in the reference configuration Ω.

I describes the total elastic energy of the body under deformations
(volume forces included, boundary tractions may give extra linear term).

We are interested in the equilibrium state of u
which describes the deformed configuration of the body.

As with most other branches of physics use the fundamental principle of energy minimization due to Euler.

According to this, these equilibrium states are exactly the minimizers of I.

The generalized Weierstrass-principle

Let M be a closed and convex subset of a reflexive Banach space X,
$I : X \to \mathbb{R}^* = \mathbb{R} \cup \{+\infty\}$, $I \not\equiv +\infty$ *weakly lower semicontinuous in X and*

$$I(u) \geq \varphi(\|u\|) \quad \textit{for} \quad \forall\, u \in M \quad \textit{where} \;\; \varphi \in C(\mathbb{R}^+)\,,\;\; \varphi(t) \underset{t\to\infty}{\longrightarrow} \infty\,.$$

Then there exists at least one $\tilde{u} \in M$ with $I(\tilde{u}) = \inf_{v\in M} I(v)$.

The generalized Weierstrass-principle provides:
An essential condition for the existence and consequently for the computation of minimizers of energy-functionals $I : X \to \mathbb{R}^*$ is the weak lower semicontinuity.

$$\liminf_{n\to\infty} I(u_n) \geq I(u) \quad \text{whenever} \quad u_n \rightharpoonup u \quad \text{in } X$$

The following results clarify the connection between the notions of convexity and weak lower semicontinuity in the scalar case.

Theorem 1 (necessity of convexity)

Let $\Omega \subset \mathbb{R}^d$ *be a bounded domain,* $p \in [1, \infty]$ *and* $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{\max\{d,m\}} \longrightarrow \mathbb{R}$ *a continuous function* $(m = 1$ *or* $d = 1)$ *such that there exists an integrable majorant* $g : \Omega \times \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ *to* f *with*

$$|f(x, u, v)| \leq g(x, |u|, |v|) \ , \quad g \in L^1(\Omega \times \mathbb{R}^+ \times \mathbb{R}^+)$$

for a.e. $x \in \Omega$ *and all* $u \in \mathbb{R}^m$, $v \in \mathbb{R}^{\max\{d,m\}}$. *Then the following implication holds*

$$I(u) = \int_\Omega f(x, u, \nabla u)\, dx \quad \text{is weakly lower semicontinuous in } W^{1,p}(\Omega, \mathbb{R}^m)$$

$$\Longrightarrow \quad f(x, u, \cdot) \quad \text{is convex for a.e. } x \in \Omega \quad \text{and all } u \in \mathbb{R}^m .$$

Theorem 2 (sufficiency of convexity)

Let $\Omega \subset \mathbb{R}^d$ *be a bounded domain,* $p \in [1, \infty]$ *and* $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{\max\{d,m\}} \to \mathbb{R}$ *a continuous function* $(m = 1$ *or* $d = 1)$ *bounded from below with*

$$f(x, u, v) \geq \langle\, g_1(x)\,,\, v \,\rangle_{\mathbb{R}^{\max\{d,m\}}} + g_2(x)$$

for a.e. $x \in \Omega$ *and all* $u \in \mathbb{R}^m$, $v \in \mathbb{R}^{\max\{d,m\}}$ *and for some* $g_1 \in L^{p'}(\Omega, \mathbb{R}^{\max\{d,m\}})$, $g_2 \in L^1(\Omega)$. *Then the following implication holds*

$$f(x, u, \cdot) \quad \text{is convex for a.e. } x \in \Omega \quad \text{and all } u \in \mathbb{R}^m$$

$$\Longrightarrow \quad I(u) = \int_\Omega f(x, u, \nabla u)\, dx \quad \text{is weakly lower semicontinuous in } W^{1,p}(\Omega, \mathbb{R}^m)\,.$$

The vectorial case $(m > 1$ and $d > 1)$

The convexity of f (with appropriate growth conditions) is still a sufficient condition for weak lower semicontinuity of I but it is far from being necessary. Therefore we adopt the notion of

Quasiconvexity (Morrey 1952)
An integrable function $f : \mathbb{R}^{m\times d} \to \mathbb{R}$ is said to be *quasiconvex* if

$$f(A) \le \frac{1}{|\Omega|} \int_\Omega f(A + \nabla\varphi(x))\, \mathrm{d}x \ , \ \forall\, A \in \mathbb{R}^{m\times d} \ , \ \forall\, \varphi \in W_0^{1,\infty}(\Omega, \mathbb{R}^m)\,.$$

On the analogy of the scalar case we get similar weak lower semicontinuity results where convexity of f is replaced by quasiconvexity.

However, due to the nonlocal character of quasiconvexity (Kristensen 1999), it is difficult to decide whether a function f is quasiconvex or not. Therefore one is interested in weakening resp. strengthening this notion to get at least necessary resp. sufficient conditions for weak lower semicontinuity of I which are easy to verify. With this motivation we obtain a weaker condition of

Rank-one-convexity (Morrey)
$f : \mathbb{R}^{m\times d} \longrightarrow \mathbb{R}$ is said to be *rank-one-convex* if

$$f(\lambda A + (1-\lambda)B) \le \lambda f(A) + (1-\lambda) f(B) \, , \; \forall \lambda \in [0,1] \, , \; \forall A, B \in \mathbb{R}^{m\times d}$$

with $rank(A-B) \le 1$.

and a stronger condition

Polyconvexity (Ball 1977)
For $A \in \mathbb{R}^{m\times d}$ let $T(A)$ denote the vector composed by A and all its quadratic minors.

$$T(A) \in \mathbb{R}^{\sigma(m,d)} \quad \text{with} \quad \sigma(m,d) = \sum_{l=1}^{\min(m,N)} \frac{m!\,d!}{(l!)^2\,(m-l)!\,(d-l)!}$$

A function f is called *polyconvex* if there exists a convex function $g : \mathbb{R}^{\sigma(m,d)} \longrightarrow \mathbb{R}$ such that

$$f(A) \,=\, g(T(A))\,, \;\; \forall\, A \in \mathbb{R}^{m\times d}\,.$$

As indicated above there is the following chain of implications between these notions.

$$f \text{ is convex} \implies f \text{ is polyconvex} \implies f \text{ is quasiconvex} \implies f \text{ is rank-one-convex}$$

These implications are not invertible in the vectorial case.
See e.g. [Ball, Sverak, Dacorogna] for the second and third one. In the scalar case, the rank-one-condition in the definition of rank-one-convexity makes no restriction to classical convexity. Hence these notions coincide and we obtain an equivalence between the convexity-classes if $m = 1$ or $d = 1$.

Polyconvexity in nonlinear elasticity

The polyconvexity condition is sufficient for quasiconvexity and hence for weak lower semicontinuity of the according functional I. In nonlinear elasticity we have $m = d = 3$ frequently. Thus $f : \mathbb{R}^{3\times 3} \longrightarrow \mathbb{R}$ is polyconvex if there exists a convex $g : \mathbb{R}^{19} \longrightarrow \mathbb{R}$ with

$$f(A) \, = \, g\left(A, \mathrm{adj} A, \det A\right) \, , \; \forall \, A \in \mathbb{R}^{3\times 3} \, .$$

recall:adjugate = transpose of cofactor matrix

Some simple examples for polyconvex functions are

(a) $f(A) \, = \, \det A$, which is an example for a polyconvex function that is not convex,

(b) $f(A) \, = \, \langle \, A \, , \, B \, \rangle^2_{\mathbb{R}^{3\times 3}} \, := \, \big(\mathrm{tr}(A \, B^T)\big)^2$ for a fixed $B \in \mathbb{R}^{3\times 3}$

There is a wide range of material models which posses a polyconvex stored energy function. Well known and often used models are

$$\begin{aligned} \text{Neo-Hookean materials} \quad &: \quad f(A) \; = \; a_1 \, \|A\|^2_{\mathbb{R}^{3\times 3}} + h(\det A) \\ \text{Mooney-Rivlin materials} \quad &: \quad f(A) \; = \; a_1 \, \|A\|^2_{\mathbb{R}^{3\times 3}} + a_2 \, \|\text{adj}\, A\|^2_{\mathbb{R}^{3\times 3}} + h(\det A) \end{aligned}$$

for some $a_1, a_2 > 0$ and a convex function $h : \mathbb{R} \to \mathbb{R}$.

On the other hand there are many material models which do not have a polyconvex energy such as the

Simo/Ortiz - energy:

$$f(A) = \frac{1}{a_1} \|A\|^{a_1}_{\mathbb{R}^{3\times 3}} + \ln(\det A)^{a_2} - \ln(\det A) \quad \text{for } a_1, a_2 \geq 2\,.$$

So we have lost the pointwise sufficient condition for quasiconvexity. This is motivating for a further generalization of the convexity notion.

Minimization of non(quasi) convex integrands

drawbacks of quasiconvexity:

- not applicable in important nonlinear material models of elasticity
- only necessary for weak lower semicontinuity

So abandon weak lower semicontinuity

This needs:
Alternative methods which characterize the minimum state of I.

An advantageous method:
Approach of Young measures (parametrized measures)

There may be situations where we need to identify $\lim_{n \to \infty} I(u_n)$ for an oscillatory sequence (u_n) which does not minimize but only infimize the functional I. This is due to the lack of weak lower semicontinuity of I. Therefore we are led to introduce a parametrized measure ν_x , $x \in \Omega$ generated by (u_n) which describes - in contrast to the weak limit of (u_n) - the minimum state of I.

Examples

(a) Set $\Omega = (0, \pi)$, $u_j(x) = \cos(j\,x)$ and $f(z) = z^2$ for $z \in [-1, 1]$ (f is continuously extended to zero elsewhere) then

$$u_j \overset{*}{\rightharpoonup} 0 \quad \text{and} \quad f(u_j) \overset{*}{\rightharpoonup} \frac{1}{2} \quad \text{in} \quad L^\infty(\Omega, \mathbb{R})\,.$$

(b) Sensitized with this result, we examine another sequence $u_j : \mathbb{R} \to \mathbb{R}$ with

$$u_j(x) \;=\; \mathbb{I}_{\{(k,k+1/3)\,,\,k\in\mathbb{Z}\}}(j\,x) = \begin{cases} 1 \;, & 0 < j\,x - \lfloor j\,x \rfloor < 1/3 \\ 0 \;, & 1/3 < j\,x - \lfloor j\,x \rfloor < 1 \end{cases}$$

more closely. The weak - $*$ - limit of (u_j) in $L^\infty(\mathbb{R})$ is the mean value $1/3$.

But if we look at the asymptotic behaviour of $f(u_j)$, we can see that

$$f(u_j) \stackrel{*}{\rightharpoonup} \frac{1}{3} f(1) + \frac{2}{3} f(0)$$

which is not equal to $f(1/3)$ for nonlinear f in general.

But how does the weak limit look like?

Theorem 3 (Existence of Young measures generated by measurable sequences)

Let $\Omega \subset \mathbb{R}^d$ be a Lebesgue measurable and bounded set, $u_j : \Omega \to \mathbb{R}^m$, $j \in \mathbb{N}$ a sequence of Lebesgue measurable functions and $f \in C_0(\mathbb{R}^m)$.
Then $(f(u_j))_{j\in\mathbb{N}}$ possesses a weak -$*$- convergent subsequence in $L^\infty(\Omega)$ and its weak -$*$- limit can be identified with the duality product between f and a Lebesgue measurable family of Radon measures (named Young measures) with finite mass $(\nu_x)_{x\in\Omega} \subset \mathcal{M}(\mathbb{R}^m)$.

$$f(u_j) \stackrel{*}{\rightharpoonup} \langle f , \nu_x \rangle \quad \text{in} \quad L^\infty(\Omega)$$

Remark

In contrast to the preceding theorem, the following one only states the existence of a Young measure as a representation of the weak limit in the case "$(f(u_j))_{j\in\mathbb{N}}$ converges weakly in $L^p(\Omega)$". In applications to minimization problems, the convergence itself has to be shown separately; depending on the specific structure of the variational problem.

Theorem 4 (Extension to "$f \in C(\mathbb{R}^m)$")

Let $\Omega \subset \mathbb{R}^d$ *be a measurable set and* $(u_j)_{j\in\mathbb{N}} \subset L^q(\Omega, \mathbb{R}^m)$,
$\sup_{j\in\mathbb{N}} \|u_j\|_{L^q(\Omega,\mathbb{R}^m)} < \infty$ *a uniformly bounded sequence for some* $q \in [1, \infty]$.
Moreover, let $f : \mathbb{R}^m \to \mathbb{R}$ *be a continuous function, such that the the sequence* $(f(u_j))_{j\in\mathbb{N}}$ *is weakly convergent in* $L^p(\Omega)$, $p \in [1, \infty]$.
Then there exists a measurable family $(\nu_x)_{x\in\Omega} \subset (C_0(\mathbb{R}^m))^*$ *(Young measure) which represents the weak limit of* $(f(u_j))_{j\in\mathbb{N}}$ *with*

$$\lim_{j\to\infty} \int_\Omega f(u_j(x))\, \alpha(x)\, dx = \int_\Omega \langle\, f\, ,\, \nu_x \,\rangle\; \alpha(x)\, dx\, , \quad \forall\, \alpha \in L^{p'}(\Omega)\, .$$

$\mathcal{A}$ - convexity and $\mathcal{A}$ - convex relaxations

To absorb restrictions of variational problems for infimizing sequences, we consider special classes of sequences in $L^q(\Omega, \mathbb{R}^m)$, $q \in [1, \infty]$ and denote them with $\mathcal{A}$. By theorem 4, any uniformly bounded sequence in $L^q(\Omega, \mathbb{R}^m)$ contains a subsequence that generates a Young measure $(\nu_x)_{x\in\Omega} \subset C_0(\mathbb{R}^m)$. Therefore - to guarantee the existence of Young measures - we require

$$\mathcal{A} \subset \left\{ (u_j)_{j\in\mathbb{N}} \subset L^q(\Omega, \mathbb{R}^m) \; ; \; \sup_{j\in\mathbb{N}} \|u_j\|_{L^q(\Omega,\mathbb{R}^m)} < \infty \right\} \subset \left(L^q(\Omega, \mathbb{R}^m)\right)^{\mathbb{N}}$$

$$q \in [1, \infty] \,.$$

According to these classes we construct a generalized notion of convexity - $\mathcal{A}$ - convexity - with the help of Young measures. At first we introduce the set of homogeneus Young measures ($\nu_x = \nu$ for almost every $x \in \Omega$) generated by sequences in $\mathcal{A}$.

$$\begin{aligned}
\mathcal{A}^* &= \Big\{ \nu \in (C_0(\mathbb{R}^m))^* \;;\; \nu \text{ is generated by a sequence in } \mathcal{A} \Big\} \\
&= \Big\{ \nu \in (C_0(\mathbb{R}^m))^* \;;\; \forall f \in C(\mathbb{R}^m)\ \exists\, (u_j) \in \mathcal{A} : \langle\, f \,,\, \nu \,\rangle \\
&\qquad = \lim_{j \to \infty} \frac{1}{|\Omega|} \int_\Omega f(u_j(x))\, \mathrm{d}x \Big\}
\end{aligned}$$

Now we are in the position to introduce the $\mathcal{A}$- convex relaxation of an arbitrary continuous function $f : \mathbb{R}^m \longrightarrow \mathbb{R}$.

$$f^{\mathcal{A}}(z) := \inf_{(u_j)\in\mathcal{A}} \left\{ \liminf_{j\to\infty} \frac{1}{|\Omega|} \int_\Omega f(u_j(x))\, \mathrm{d}x \,;\, u_j \rightharpoonup z \text{ in } \mathcal{A} \right\} \;;\;\; z \in \mathbb{R}^m$$

where the convergence " $u_j \rightharpoonup u$ in $\mathcal{A}$ " depends on the choice of $\mathcal{A}$.

With the usage of Young measure notation we obtain

$$f^{\mathcal{A}}(z) = \inf_{\nu \in \mathcal{A}^*} \{\langle f , \nu \rangle ; \langle id , \nu \rangle = z\} , \quad \langle f , \nu \rangle = \int_{\mathbb{R}^m} f(\tilde{z}) \, \mathrm{d}\nu(\tilde{z}) .$$

Definition

A Lebesgue measurable function $f : \mathbb{R}^m \longrightarrow \mathbb{R}$ *is called* $\mathcal{A}$*-convex if it is equal to its* $\mathcal{A}$*-convex relaxation* $f^{\mathcal{A}}$.

Example 1

We consider

$$\mathcal{A} = \left\{ (\nabla u_j)_{j\in\mathbb{N}} \,:\, (u_j)_{j\in\mathbb{N}} \subset W^{1,q}(\Omega,\mathbb{R}^m)\,,\ \sup_{j\in\mathbb{N}} \|u_j\|_{W^{1,q}(\Omega,\mathbb{R}^m)} < \infty \right\}.$$

The sequence of gradients $(\nabla u_j)_{j\in\mathbb{N}}$ is a sequence of matrices in $\mathbb{R}^{m\times d}$.
$\mathcal{A}$- convexity corresponds to the previously introduced Quasiconvexity in this case.

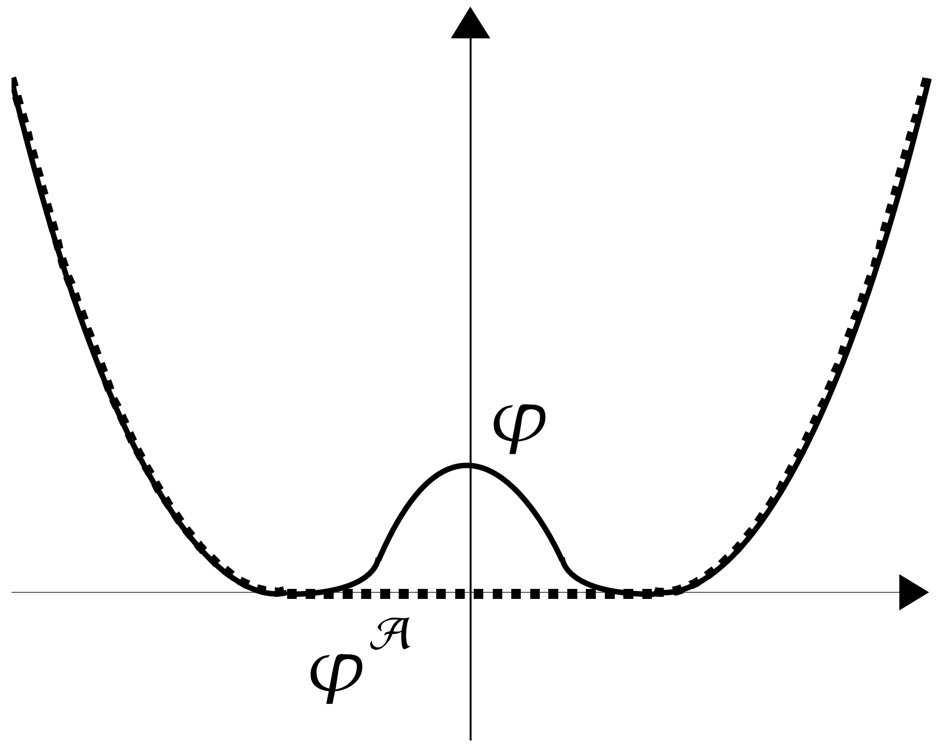
$$f \text{ is } \mathcal{A}\text{- convex} \quad \Longleftrightarrow \quad f(A) \le \frac{1}{|\Omega|}\int_\Omega f(A+\nabla\varphi(x))\,\mathrm{d}x \ ,\ \forall\, A \in \mathbb{R}^{m\times d}, \\ \forall\, \varphi \in C_0^\infty(\Omega,\mathbb{R}^m)$$

Example 2

We compute a $\mathcal{A}$-convex relaxation of a non-$\mathcal{A}$- convex function φ explicitly. To this end we set

$$\Omega = (0,1) \quad \text{and} \quad \mathcal{A} = \left\{ (u_j)_{j\in\mathbb{N}} \subset L^2((0,1)) \; ; \; \exists\, C > 0 \, : \, \sup_{j\in\mathbb{N}} \|u_j\|_{L^2((0,1))} \leq C \right\} .$$

Consider $\varphi : \mathbb{R} \to \mathbb{R}$ with $\varphi(z) = z^4 - z^2 + \frac{1}{4}$.



Then

$$\varphi^{\mathcal{A}}(z) = \begin{cases} 0 \ , & z \in \left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right] \\ z^4 - z^2 + \frac{1}{4} \ , & z \in \mathbb{R} \setminus \left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right] \end{cases}$$

Application to unilateral contact – basics

Let Ω bdd domain $\subset \mathbb{R}^3$ with Lipschitz bdy $\Gamma = \overline{\Gamma}_c \cup \overline{\Gamma}_D \cup \overline{\Gamma}_N$

unknown $u \in \mathcal{W}^{1,p} := \left[W^{1,p}(\Omega)\right]^3 = W^{1,p}(\Omega, \mathbb{R}^3)$

$$X = \left\{u, H, \tau) : H = \text{adj } \nabla u, \tau = \det \nabla u\right\}$$

recall:adjugate = transpose of cofactor matrix

Consider $X \subset \mathcal{X} := \mathcal{W}^{1,p} \times \mathcal{L}^q \times L^r, \quad p, q, r$ later

Then (J. Ball 1977)
X is not convex, but sequently weakly closed $\subset \mathcal{X}$.

Further conditions in nonlinear elasticity

The deformation u of the body should be (locally) invertible

Hence set of admissible deformations

$$V = \Big\{ u : (u, H, \tau) \in X, \tau = \det \nabla u > 0 \text{ a.e. in } \Omega \Big\}$$

May be further restricted by Dirichlet bd cond.: $u|\Gamma_D = u^0$

Loads give rise to continuous linear forms in u :

$$\begin{aligned} (f, u) &= \int_{\Omega} f \cdot u \, dx, \;\; f \ldots \text{ volume force density} \\ \langle g, u \rangle &= \int_{\Gamma_N} g \cdot u \, da, \;\; g \ldots \text{ surface force density} \end{aligned}$$

The total energy

given by

$$I(u) = \int_\Omega e(x, \nabla u)\, dx - (f, u) - \langle g, u \rangle$$

with nonconvex strain energy function $e(x, \dot{)}$.

Assume $e(x, \cdot)$ *polyconvex*:

$\exists\ \tilde{e}$: $\Omega \times \mathbb{R}^{3\times 3} \times \mathbb{R}^{3\times 3} \times \mathbb{R}_{++} \to \mathbb{R}$ such that
$\forall\ x \in \Omega$ $\tilde{e}(x, \cdot, \cdot, \cdot)$ is convex
$\forall\ M \in \mathbb{R}^{3\times 3}$ with $\det M > 0$:

$$e(x, M) = \tilde{e}(x, M, \text{adj } M, \det M)$$

Further assumptions on e

Assume $\tilde{e}$ *coercive:*

$$\exists\, a \in L^1(\Omega); b > 0, p \geq 2, q \geq \frac{p}{p-1}, r > 1$$

such that for all $x \in \Omega, M \in \mathbb{R}^{3\times 3}, H \in \mathbb{R}^{3\times 3}, \tau > 0$

$$\tilde{e}(x, M, H, \tau) \geq a(x) + b(\|M\|^p + \|H\|^q + \tau^r)$$

These assumptions are satisfied in various examples of nonlinear material models:

Ogden model, Mooney-Rivlin model

Semicoercive contact problem

If Dirichlet by cond. absent, only Neumann by cond. (surface force) and Signorini cond. (rigid obstacle). In this situation

Proposition 1. *Let* $|u|_{1,p} := (\int_\Omega |\nabla u|^p dx)^{1/p}$. *Then*

(i) $\|\cdot\|_{1,p} \cong |\cdot|_{1,p} + \|\cdot\|_p$

(ii) $\mathcal{R} := \{\varrho \in \mathcal{W}^{1,p}, |\varrho|_{1,p} = 0\}$
$= \{\varrho : \Omega \to \mathbb{R}^3$ *const.* $\} \cong \mathbb{R}^3$ *of finite dimension*

(iii) $\exists\, c_0 > 0 \;\forall\, u \in \mathcal{W}^{1,p}$
$\inf\{\|u - \varrho\|_p : \varrho \in \mathcal{R}\} \leq c_0 |u|_{1,p}$

Then Variational Problem for given $K \neq \emptyset$ closed convex $\subset \mathbb{R}^3$

(P) Find a minimizer of $I(u)$ on V subject to $u(x) \in K$ for (a.e.) $x \in \Gamma_c$.

Recession analysis

Asymptotic cone $ac\,K = \bigcap_{t>0} t(K - z)$, $z \in K$ fixed

Directions $d \in acK \cap \mathcal{R}$ of "escape" should form an obtuse angle with the applied forces

$$(H) \qquad \int_{\Omega} f \cdot d \, dx + \int_{\Gamma_c} g \cdot d \, da < 0, \, \forall d \in ac\,K \cap \mathcal{R} \backslash \{0\}$$

Theorem 1. [Ciarlet & Nečas,1985] *Suppose (H). Then* $\exists$ *sol. to (P).*

A penalty approximation

Now explicit constraint $j\big(u(x)\big) \le 0\,(x \in \Gamma_c)$,
where $j : \mathbb{R}^3 \to \mathbb{R}$ pos. homog., convex.

Thus feasible set

$$\hat{V} = \Big\{u \in V : j\big(u(x)\big) \le 0 \ \text{ a.e. on } \Gamma_c\Big\}$$

Introduce

$$J(u) = \beta \Big\| \big(j \circ u|\Gamma_c\big)^+ \Big\|_{L^s(\Gamma_c)} \,(\beta > 0, s \ge 1)$$

Then $J : \mathcal{W} \supset V \to \mathbb{R}$ satisfies

$$(H1) \qquad \begin{cases} \text{pos. homog., convex, } \ge 0 \\ \text{lower semicontinuous} \\ u \in V \text{ and } J(u) = 0 \Leftrightarrow u \in \hat{V} \end{cases}$$

A penalty approximation - continued

Introduce

$$I_\gamma(u) := I(u) + \gamma J(u),\ \gamma > 0.$$

Consider instead of (P)

$$(P_\gamma) \qquad \text{Find a minimizer of } I_\gamma(\cdot) \text{ on } V$$

Instead of (H) assume

$$(H2) \qquad \gamma J(\varrho) - (f, \varrho) - \langle g, \varrho \rangle > 0,\ \forall\ \varrho \in \mathcal{R} \backslash \{0\}$$

analogous to linear elasticity (Duvaut & Lions, '76)

Then (H2) $\Rightarrow$ (H) holds.

Convergence of penalty method

Theorem 2. [J.G., 1989] *Suppose J satisfies (H1) and (H2) for some $\gamma_0 > 0$.*

Let $2 \leq p < 3, q \geq \dfrac{p}{p-1}, r > 1, p \leq s \leq \dfrac{2p}{3-p}$.

Then:

$(\forall\ \gamma \geq \gamma_0)\ \exists$ *solution $u^{(\gamma)}$ to (P_γ).*

For any sequence $\gamma_\nu \to \infty$ we have

$u^{(\gamma_\nu)} \rightharpoonup u$ *a solution to (P).*

Exactness of penalty method

Proposition 2. *Suppose* $\exists$ *sol.* $\overline{u}$ *to (P), further suppose*

$$\exists\, \overline{\lambda} \in L^{s'}(\Gamma_c), \overline{\lambda} \geq 0 \text{ a.e. } \left(\frac{1}{s'} + \frac{1}{s} = 1\right):$$

$$(\forall\, v \in V) I(\overline{u}) \leq I(v) + \int_{\Gamma_c} \overline{\lambda} \cdot (j \circ v) da$$

$$\text{Let } \overline{\gamma} := \beta^{-1} \|\overline{\lambda}\|_{L^{s'}}$$

Then for $\gamma > \overline{\gamma}$, *any solution* $u^{(\gamma)}$ *to* (P_γ) *solves original problem (P).*

$\longrightarrow$ open problem:
existence of such a Lagrange multiplier $\overline{\lambda}$

Euler-Lagrange eq. in nonlinear unilateral contact

Partial answer by recent work of Schuricht (2002, 2006).

Refined model, constraint no longer assumed to be convex.

Instead constraint

$$g(u) \leq 0,$$

with g merely locally Lipschitz on $\mathcal{W}^{1,p}(\Omega)$.

use of Clarke's calculus of generalized convexity.

Construction of locally Lipschitz constraint

Geometric contact condition

$$u(x) \in \overline{\mathbb{R}^3 \setminus \mathcal{C}} \text{ on } \overline{\Omega}$$

where given obstacle $\mathcal{C}$ is closure of open set $\subset \mathbb{R}^3$.

Introduce signed distance function

$$\delta(q) = \text{dist}_{\mathbb{R}^3 \setminus \mathcal{C}} q - \text{dist}_{\mathcal{C}} q, \; q \in \mathbb{R}^3 .$$

Then

$$\delta(q) \leq 0 \qquad \Leftrightarrow \qquad q \in \overline{\mathbb{R}^3 \setminus \mathcal{C}}.$$

Thus constraint

$$g(u) := \max_{x \in \overline{\Omega}} \delta\big(u(x)\big) \leq 0 .$$

The Nonconvex Variational Problem

Let as above

$$I(u) = \int_\Omega e\big(x, \nabla u(x)\big) dx.$$

However

$$e(x, M) = \infty \text{ if } \det M \leq 0$$

(Deformations should be locally invertible and orientation preserving) is dropped.

Variational problem:

Find minimizer of I in $\mathcal{W}^{1,p}(\Omega; \mathbb{R}^3)$
such that $u = u^0$ on $\Gamma_D, g(u) \leq 0$.

Further assumptions

Differentiability assumptions on strain energy $e(x, \cdot)$:

$$e(x, \cdot) \text{ cont. diff. on } \mathbb{R}^{3\times 3} (\forall\, x \in \Omega);$$
$$\exists\, c_0 \geq 0, c \in L^1(\Omega) \text{ such that } \forall\, M \in \mathbb{R}^{3\times 3}, x \in \Omega$$
$$\|\nabla_M e(x, M)\| \leq c_0 \|M\|^p + c(x).$$

Hence I is differentiable in $\mathcal{W}^{1,p}$.

Moreover

$$u^0(\Gamma_D) \cap \mathcal{C} = \emptyset,$$

i.e. Dirichlet boundary cond. is compatible with contact constraint.

Euler-Lagrange equation – The result

Theorem 3. [Schuricht '02] *Let $u \in \mathcal{W}^{1,p}(\omega, \mathbb{R}^3), p > 3$ be a local minimizer of variational problem above. Then under assumptions above $\exists$ finite Borel measure μ_C with support in the contact set*

$$\{x \in \mathbb{R}^3 : u(x) \in \mathcal{C}\}$$

and a function $d^* \in L^1(\overline{\omega}, \mu_c; \mathbb{R}^3)$ with

$$d^*(x) \in \partial^\circ \delta\big(u(x)\big), \ \forall\, x \in \overline{\Omega}$$

such that for all $w \in \mathcal{W}^{1,\infty}(\Omega; \mathbb{R}^3)$ with $x|\Gamma_D = 0$

$$\int\limits_{\Omega} \nabla_M e\big(x, \nabla u(x)\big) \nabla w(x)\, dx + \int\limits_{\overline{\Omega}} d^*(x) w(x)\, d\mu_c(x) = 0\,.$$