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Pseudomonotonicity and Hemivariational Inequalities
Existence and Approximation
Applications to Nonsmooth Mechanics

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Outline

I A Review on Pseudomonotonicity -
Existence, Applications, Approximations

II Existence and General Approximation Result for abstract nonlinear nonsmooth variational problems

III Smoothing approximations

IV Numerical FE approximation of nonlinear nonsmooth functionals - Applications to HVI

V Applications to Nonsmooth Mechanics

- Unilateral contact with friction
- Adhesive contact problem
- Nonmonotone skin friction problem

A Review on Pseudomonotonicity - Existence

(Topologically) Pseudomonotone Operators in Nonlinear Analysis and Nonlinear PDEs are going back to H. Brézis 1968
F.E. Browder exploited the fact that Monotone Operator + Compact Operator = Pseudomonotone (PM) in several papers on general elliptic/parabolic PDEs
F.E. Browder & P. Hess, 1972: Sum of PM is again PM
P. Hess, 1974: Semi-coercive problems
gave then a lecture at University of Mannheim

Pseudomonotone Operators

Definition 1

The operator $A : X \to X^*$ *is pseudomonotone iff* $u_n \rightharpoonup u$ *and* $\limsup_{n\to\infty}\langle Au_n, u_n - u\rangle \le 0$ *implies* $Au_n \rightharpoonup Au$ *and* $\langle Au_n, u\rangle \to \langle Au, u\rangle$.

Moreover, the following implications hold

- If A is monotone and hemicontinuous, then A is pseudomonotone.
- If A is strongly continuous, then A is pseudomonotone.

Definition 2

Let $A : X \to X^*$*. Then* A *is called*

- **strongly continuous** *iff* $u_n \rightharpoonup u$ *implies* $Au_n \to Au$.
- **hemicontinuous** *iff the real function* $t \to \langle A(u + tv), w\rangle$ *is continuous on* $[0, 1]$ *for all* $u, v, w \in X$.

A Review on Pseudomonotonicity - Existence continued

On the other hand, W. Oettli initiated the study of equilibrium problems, defined by monotone (bi)functions
J. Gw., Ph D Thesis 1978 at University of Mannheim :
Nichtlineare Variationsungleichungen mit Anwendungen
contains

- definition of pm (bi)functions including the case of pm set-valued operators with bounded values; Sum of PM is again PM
- existence theory based on KKM-Fan principle; see also J.Gw., Nonlin Anal TMA, 1981
- semicoercive existence theory; see also H.G. Jeggle, Nichtlineare Funktionalanalysis, Teubner, 1978.

A Review on Pseudomonotonicity - Applications

Monotone Operator + Compact Operator = Pseudomonotone $\Rightarrow$
Nonlinear elliptic PDEs ,see eg V. Mustonen
Fluid mechanics: Navier - Stokes eqs;
see eg. Tomarelli, Mikorski & Ochal
heat transfer with radiation, Boltzmann law see eg. T. Tiihonen
solid mechanics, eg. Karman plates
see e.g. D. Goeleven & J. Gw., 2000
more general Hemi Variational Inequalities (HVI)
P.D. PANAGIOTOPOULOS moreover Z. Naniwiecz et al

A Review on Pseudomonotonicity - Regularization and Approximations

Sum of PM is again PM $\Rightarrow$
Regularization method in the noncoercive case
e.g. J. Gw. 1997
more recent: O. Chadli, S. Schaible, J.C. Yao 2004
with application to noncoercive HVI
Combining regularization and penalty method:
recent: B. D. Rouhani, A. A. Khan & F. Raciti 2008:
Mosco convergence of convex sets and approximation of set-valued pm operators under Hausdorff distance assumptions

Existence and General Approximation for abstract variational problems

$(V, ||\cdot||_V)$ reflexive Banach space, $K \subset V$ nonempty closed, convex cone, $\varphi : K \times K \to \mathbb{R}$

$A : V \to V^*$ linear continuous operator, $\langle Av, v\rangle_V \geq c_0||v||_V^2$

Problem π : Find $\tilde{u} \in K$ such that

$$\langle A\tilde{u} - g, v - \tilde{u}\rangle_V + \varphi(\tilde{u}, v) \geq 0 \quad \forall v \in K.$$

Problem π_t : Find $\tilde{u}_t \in K_t$ such that

$$\langle A\tilde{u}_t - g, v_t - \tilde{u}_t\rangle_V + \varphi_t(\tilde{u}_t, v_t) \geq 0, \forall v_t \in K_t.$$

Assumptions

(H1) $\varphi(\cdot,\cdot)$ is a pseudomonotone functional on $K \times K$

(H2) $\varphi(u,u) \geq 0, \ \forall u \in K$

(H3) $\varphi(u,\cdot)$ is convex on K

(H4) $\varphi(\cdot,u)$ is upper semicontinuous on K

(H5) there exists a compact subset $M \subset V$ and $\tilde{v} \in M \cap K$ such that

$$\varphi(u,\tilde{v}) < -\langle Au - g, \tilde{v} - u\rangle_V \quad \forall u \in K \backslash M$$

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Theorem 1 (**Existence theorem**)

Under conditions (H1)-(H5), the problem π has at least one solution $\tilde{u} \in M \cap K$.

Assumption

$$K_t \subset K,\ K_t \xrightarrow{M} K$$

(H6) for any nets $\{u_t\}$ and $\{v_t\}$ such that $u_t \in K_t$, $v_t \in K_t$, $u_t \rightharpoonup u$ and $v_t \to v$ we have

$$\limsup_{t \in T} \varphi_t(u_t, v_t) \le \varphi(u, v)$$

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$$\limsup_{t \in T} \varphi_t(u_t, v_t) \leq \varphi(u, v)$$

Theorem 2 (General Approximation Result)

The family $\{\tilde{u}_t\}$ of solutions to the Problem π_t is uniformly bounded in V and any weak limit point of the net $\{\tilde{u}_t\}$ is a solution to the Problem π.

Smoothing approximation

$$\hat{f}(x,\varepsilon) := \int_{\mathbb{R}^m} f(x-\varepsilon u)\Phi(u)du = \int_{\mathbb{R}^m} f(y)\Theta(x-y,\varepsilon)dy,\ (x,\varepsilon) \in \mathbb{R}^m \times \mathbb{R}_{++}$$

- $\Phi : \mathbb{R}^m \to \mathbb{R}_+$ such that $\int_{\mathbb{R}^m} \Phi(u)du = 1$
- $\Theta : \mathbb{R}^m \times \mathbb{R}_{++} \to \mathbb{R}_+,\ \ \Theta(x,\varepsilon) := \varepsilon^{-m}\Phi(\varepsilon^{-1}x)$

$$\mathbb{R}_+ = \{\varepsilon\ :\ \varepsilon \geq 0,\ \varepsilon \in \mathbb{R}\}$$

$$\mathbb{R}_{++} = \{\varepsilon\ :\ \varepsilon > 0,\ \varepsilon \in \mathbb{R}\}$$

1. $f : \mathbb{R}^m \to \mathbb{R}$ is locally Lipschitz and $\operatorname{supp}\Phi$ is bounded;
   Steklov averaged function

$$\Phi(u) = \begin{cases} 1 & \text{if } \max_i |u_i| \leq 0.5 \\ 0 & \text{otherwise} \end{cases}$$

2. $f : \mathbb{R}^m \to \mathbb{R}$ is globally Lipschitz, $\operatorname{supp}\Phi$ is infinite, but Φ has to be of finite absolute mean; that is

$$\int_{\mathbb{R}^m} |u|\Phi(u)\, du < +\infty$$

   Bell function

$$\Phi(u) = \frac{1}{(\sqrt{2\pi})^m} e^{-\frac{|u|^2}{2}}$$

3. $f : \mathbb{R}^m \to \mathbb{R}$ is a max (min) function of continuously differentiable functions $\Rightarrow \hat{f}(x, \varepsilon)$ **can be explicitly expressed**;

Properties of $\hat{f}(x, \varepsilon)$

Let f be a locally Lipschitz function and Φ be a continuosly differentiable with bounded support. Then

(i1) $\hat{f}(\cdot, \cdot)$ is continuously differentiable on $\mathbb{R}^m \times \mathbb{R}_{++}$;

(i2) $\hat{f}(\cdot, \cdot)$ is locally Lipschitz continuous on $\mathbb{R}^{m+1}$;

(i3) $\lim_{\substack{z \to x \\ \varepsilon \downarrow 0}} \nabla_x \hat{f}(z, \varepsilon) \subseteq \partial f(x)$,

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$$\nabla_x \hat{f}(z, \varepsilon) = \int_{\mathbb{R}^m} f(y) \nabla_x \Theta(z - y, \varepsilon)\, dy$$

$$\partial f(x) := \{\xi \in \mathbb{R}^m \; : \; \xi^T u \leq f^0(x; u) \quad \text{for} \quad \forall u \in \mathbb{R}^m\}$$

$$f^0(x; u) := \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + tu) - f(u)}{t}.$$

Maximum-type functions

$f(x) = \max\{g_1(x), g_2(x), \dots, g_k(x)\}$

$g_i : \mathbb{R}^m \to \mathbb{R}$ continuously differentiable

According to Bertsekas

$$\begin{aligned} f(x) = \quad & g_1(x) + p(g_2(x) - g_1(x) + \dots \\ + \quad & p(g_{k-1}(x) - g_{k-2}(x) + p(g_k(x) - g_{k-1}(x)) \dots), \end{aligned}$$

where $p : \mathbb{R} \to \mathbb{R}_+$ is the plus function i.e. $p(t) = \max\{t, 0\}$.

Smoothing function (Chen, Qi and Sun)

$$\begin{aligned} S(x,\varepsilon) := g_1(x) &+ \hat{p}(\varepsilon, g_2(x) - g_1(x) + \hat{p}(\varepsilon, g_3(x) - g_2(x) + \dots \\ &+ \hat{p}(\varepsilon, g_{k-1}(x) - g_{k-2}(x) + \hat{p}(\varepsilon, g_k(x) - g_{k-1}(x)) \dots), \end{aligned}$$

where

$$\hat{p}(\varepsilon, t) = \int_{\mathbb{R}} p(t - \varepsilon s)\phi(s)\, ds$$

and

$$\kappa := \int_{\mathbb{R}} |t|\Phi(t)\, dt < +\infty.$$

Properties of $\hat{p}(\varepsilon, t)$

- For any $\varepsilon > 0$ and $t \in \mathbb{R}$,

$$|\hat{p}(\varepsilon, t) - p(t)| \leq \kappa\varepsilon;$$

- For any $\varepsilon > 0$, $\hat{p}(\varepsilon, \cdot)$ is continuously differentiable on $\mathbb{R}$,

$$\hat{p}'_t(\varepsilon, t) = \int_{-\infty}^{\frac{t}{\varepsilon}} \Phi(s)\, ds$$

and

$$\hat{p}'_t(\varepsilon, t) \in [0, 1].$$

Moreover, if $\operatorname{supp} \Phi = \mathbb{R}$ then $\hat{p}'_t(\varepsilon, t) \in (0, 1)$.

Properties of $S(x, \varepsilon)$

- For any $\varepsilon > 0$ and $x \in \mathbb{R}^m$, $|S(x, \varepsilon) - f(x)| \leq (k-1)\kappa\varepsilon$;
- $\nabla_x S(x, \varepsilon) = \sum_{i=1}^{k} \Lambda_i \nabla g_i(x), \quad \text{where } \Lambda_i \in [0,1], \ \sum_{i=1}^{k} \Lambda_i = 1;$
- $\lim_{\substack{z \to x \\ \varepsilon \downarrow 0}} \nabla_x S(z, \varepsilon) \subseteq \partial f(x).$

Neural networks smoothing function

If we choose

$$\Phi(t) = \frac{e^{-t}}{(1+e^{-t})^2} \quad \Rightarrow$$

then

$$\hat{p}(\varepsilon, t) = t + \varepsilon \ln(1 + e^{-\frac{t}{\varepsilon}}), \quad (\varepsilon, t) \in \mathbb{R}_{++} \times \mathbb{R}.$$

and

$$\hat{f}(x, \varepsilon) := S(x, \varepsilon) = \varepsilon \ln \Big(\sum_{i=1}^{k} e^{g_i(x)/\varepsilon} \Big), \; (x, \varepsilon) \in \mathbb{R}^m \times \mathbb{R}_{++}.$$

Zang smoothing function

If we choose

$$\rho(t) = \begin{cases} t & \text{if } -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

then

$$\hat{p}(\varepsilon, t) = \begin{cases} 0 & \text{if} \quad t < -\frac{\varepsilon}{2} \\ \frac{1}{2\varepsilon}(t + \frac{\varepsilon}{2})^2 & \text{if} \quad -\frac{\varepsilon}{2} \leq t \leq \frac{\varepsilon}{2} \\ t & \text{if} \quad t > \frac{\varepsilon}{2} \end{cases} .$$

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If $f(x) := \max\{g_1(x), g_2(x)\} = g_1(x) + p(g_2(x) - g_1(x))$ then

$$\hat{f}(x, \varepsilon) = \begin{cases} g_1(x) & \text{if } (i) \text{ holds} \\ \frac{1}{2\varepsilon}[g_2(x) - g_1(x)]^2 + \frac{1}{2}(g_2(x) + g_1(x)) + \frac{\varepsilon}{8} & \text{if } (ii) \text{ holds} \\ g_2(x) & \text{if } (iii) \text{ holds} \end{cases}$$

where

$$\begin{array}{ll} (i) & g_2(x) - g_1(x) \leq -\frac{\varepsilon}{2} \\ (ii) & -\frac{\varepsilon}{2} \leq g_2(x) - g_1(x) \leq \frac{\varepsilon}{2} \\ (iii) & g_2(x) - g_1(x) \geq \frac{\varepsilon}{2}. \end{array}$$

Applications to HVI on boundary

$V = H^1(\Omega; \mathbb{R}^m)$, $\Omega \subset \mathbb{R}^2$ is a plane polygonal domain with boundary Γ

$f : \mathbb{R}^m \to \mathbb{R}$ is a locally Lipschitz function such that

(A1) for each $\eta \in \mathbb{R}^m$, $\eta^* \in \partial f(\eta) \Rightarrow |\eta^*| \leq c(1 + |\eta|)$

(A2) for each $\eta \in \mathbb{R}^m$, $\eta^* \in \partial f(\eta)$, $\eta^* \cdot (-\eta) \leq b\,|\eta|$ i.e.

$$\begin{cases} \eta^* \geq -b & \text{if} \quad \eta > 0 \\ \eta^* \leq b & \text{if} \quad \eta < 0 \end{cases}$$

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Problem (P): Find $u \in K$ such that

$$\langle Au - g, v - u \rangle_V + \int_\Gamma f^0(\gamma u; \gamma v - \gamma u)\, ds \geq 0, \quad \forall v \in K,$$

where $\gamma : H^1(\Omega; \mathbb{R}^m) \to L^2(\Gamma; \mathbb{R}^m)$ is a trace mapping.

Piecewise linear approximations. Finite dimensional discretization

Approximate V by

$$V_h^1 = \{v_h \in C(\overline{\Omega}; \mathbb{R}^m) : v_h|_\Delta \in (P_1)^m, \forall \Delta \in \mathcal{T}_h\}$$

and K by a family K_h of closed convex nonempty cones of V_h^1 such that $K_h \xrightarrow{M} K$; We put $V_{n,h}^1 := V_h^1,\ K_{n,h} := K_h$;

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$\Pi =$ linear continuous mapping transforming vector valued functions into scalar ones, for example

$$\Pi y := y_j|_\Gamma \quad \text{for some given } j \in \{1, \dots, m\}$$

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$$\Pi y := y_j|_\Gamma \quad \text{for some given } j \in \{1, \ldots, m\}$$

$$W_h^1 = \Pi(V_h^1),\ W_h^1 = \{w_h \in C(\Gamma) : w_h|_E \in P_1,\ \forall E \in \tilde{\mathcal{T}}_h\}$$

$$\tilde{\mathcal{T}}_h = \{ \text{ set of all edges of the boundary triangles } \Delta \in \mathcal{T}_h\}$$

Approximation by smoothing function

Denote

$$\varphi(u, v) := \int_{\Gamma} f^0(\gamma u; \gamma v - \gamma u)\, ds.$$

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We define $\varphi_n : u \in V \rightsquigarrow \varphi_n(u) \in V^*$ by

$$\langle \varphi_n(u), v\rangle = \int_\Gamma \nabla_x \hat{f}(\gamma u, \varepsilon_n) \cdot \gamma v \, ds$$

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Approximate $\nabla_x \hat{f}(\gamma u_{n,h}, \varepsilon_n) \cdot \gamma v_{n,h}$ by $w_{n,h}(u_{n,h}, v_{n,h})$ via:

(1) $w_{n,h}(u_{n,h}, v_{n,h}) \in W^1_{n,h}$, $W^1_{n,h} := W^1_h$

(2) $w_{n,h}(u_{n,h}, v_{n,h})(P) = \nabla_x \hat{f}(\gamma u_{n,h}(P), \varepsilon_n) \cdot \gamma v_{n,h}(P)$ for $\forall P \in \Gamma \cap \Sigma_h$, where Σ_h is the set of the vertices of $\mathcal{T}^h$.

Using Kepler's trapezoidal rule, approximate $\langle \varphi_n(u_{n,h}), v_{n,h} \rangle$ by

$$\begin{aligned} \langle \varphi_{n,h}^{(1)}(u_{n,h}), v_{n,h} \rangle &= \int_\Gamma w_{n,h}(u_{n,h}, v_{n,h})\, ds \\ &= \frac{1}{2} \sum_i |P_i P_{i+1}| \big[\nabla_x \hat{f}(\gamma u_{n,h}(P_i), \varepsilon_n) \cdot \gamma v_{n,h}(P_i) \\ &+ \nabla_x \hat{f}(\gamma u_{n,h}(P_{i+1}), \varepsilon_n) \cdot \gamma v_{n,h}(P_{i+1}) \big]. \end{aligned}$$

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$$\begin{aligned} \langle \varphi^{(1)}_{n,h}(u_{n,h}), v_{n,h} \rangle &= \int_\Gamma w_{n,h}(u_{n,h}, v_{n,h})\, ds \\ &= \frac{1}{2} \sum_i |P_i P_{i+1}| \left[\nabla_x \hat{f}(\gamma u_{n,h}(P_i), \varepsilon_n) \cdot \gamma v_{n,h}(P_i) \right. \\ &+ \left. \nabla_x \hat{f}(\gamma u_{n,h}(P_{i+1}), \varepsilon_n) \cdot \gamma v_{n,h}(P_{i+1}) \right]. \end{aligned}$$

Problem $(P_{n,h})$: find $u_{n,h} \in K_{n,h}$ such that

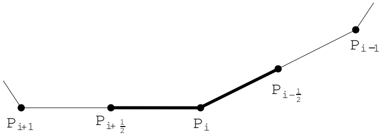
$$\langle A u_{n,h} - g, v_{n,h} - u_{n,h} \rangle_V + \varphi_{n,h}(u_{n,h}, v_{n,h}) \geq 0, \quad \forall v_{n,h} \in K_{n,h}$$

$$\varphi_{n,h}(u_{n,h}, v_{n,h}) := \langle \varphi^{(1)}_{n,h}(u_{n,h}), v_{n,h} - u_{n,h} \rangle$$

We apply the General Approximation Result to *Problem* $(P_{n,h})$.

Define $q_h : C(\Gamma; \mathbb{R}^m) \to L^\infty(\Gamma; \mathbb{R}^m)$ by

$$q_h(\mu) = \sum_{P_i \in \Gamma \cap \Sigma_h} \mu(P_i)\, \chi_i, \quad \forall \mu \in C(\Gamma; \mathbb{R}^m)$$



$$\chi_i = \chi_{(P_{i-\frac{1}{2}}, P_{i+\frac{1}{2}})}$$

Then

$$\int_\Gamma w_{n,h}(u_{n,h}, v_{n,h})\, ds = \int_\Gamma \nabla_x \hat{f}(q_h(\gamma u_{n,h}), \varepsilon_n) \cdot q_h(\gamma v_{n,h})\, ds.$$

Piecewise quadratic approximations

Approximate V by

$$V_h^2 = \{v_h \in C(\overline{\Omega}; \mathbb{R}^m) : v_h|_\Delta \in (P_2)^m, \forall \Delta \in \mathcal{T}^h\}$$

and define $\quad W_h^2 = \{w_h \in C(\Gamma) : w_h|_E \in P_2, \; \forall E \in \tilde{\mathcal{T}}_h\}.$

Approximate $\nabla_x \hat{f}(\gamma u_{n,h}, \varepsilon_n) \cdot \gamma v_{n,h}$ by $w_{n,h}(u_{n,h}, v_{n,h})$ via:

(1) $w_{n,h}(u_{n,h}, v_{n,h}) \in W_{n,h}^2, \; W_{n,h}^2 := W_h^2$

(2) $w_{n,h}(u_{n,h}, v_{n,h})(P) = \nabla_x \hat{f}(\gamma u_{n,h}(P), \varepsilon_n) \cdot \gamma v_{n,h}(P)$
for $\quad \forall P \in \Gamma \cap (\Sigma_h \cup \Sigma_h'),$

Σ_h' is the set of all midpoints of the sides of the triangles of $\mathcal{T}^h$.

Using Simpson's rule, approximate $\langle \varphi_n(u_{n,h}), v_{n,h} \rangle$ by

$$\begin{aligned}
\langle \varphi^{(2)}_{n,h}(u_{n,h}), v_{n,h} \rangle &= \int_\Gamma w_{n,h}(u_{n,h}, v_{n,h})\, ds \\
&= \frac{1}{6} \sum_i |P_i P_{i+1}| \Big[\nabla_x \hat{f}(\gamma u_{n,h}(P_i), \varepsilon_n) \cdot \gamma v_{n,h}(P_i) \\
&+ 4 \nabla_x \hat{f}(\gamma u_{n,h}(P_{i+\frac{1}{2}}), \varepsilon_n) \cdot \gamma v_{n,h}(P_{i+\frac{1}{2}}) \\
&+ \nabla_x \hat{f}(\gamma u_{n,h}(P_{i+1}), \varepsilon_n) \cdot \gamma v_{n,h}(P_{i+1}) \Big].
\end{aligned}$$

Problem $(P_{n,h})$: find $u_{n,h} \in K_{n,h}$ such that

$$\langle A u_{n,h} - g, v_{n,h} - u_{n,h} \rangle_V + \varphi_{n,h}(u_{n,h}, v_{n,h}) \geq 0, \quad \forall v_{n,h} \in K_{n,h}.$$

$$\text{Now} \quad \varphi_{n,h}(u_{n,h}, v_{n,h}) = \langle \varphi^{(2)}_{n,h}(u_{n,h}), v_{n,h} - u_{n,h} \rangle.$$

Applications to HVI in domain

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Problem (P): Find $u \in K$ such that

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- **piecewise linear approximations**

$$V_h^1 = \{v_h \in C(\overline{\Omega}; \mathbb{R}^m) : v_h|_\Delta \in (P_1)^m, \forall \Delta \in \mathcal{T}^h\}$$

$$W_h^1 = \{w_h \in C(\overline{\Omega}) : w_h|_\Delta \in P_1, \forall \Delta \in \mathcal{T}^h\}$$

Approximate $\nabla_x \hat{f}(u_{n,h}, \varepsilon_n) \cdot v_{n,h}$ by $w_{n,h}(u_{n,h}, v_{n,h})$ defined by:

(1) $w_{n,h}(u_{n,h}, v_{n,h}) \in W^1_{n,h}$, $W^1_{n,h} := W^1_h$

(2) $w_{n,h}(u_{n,h}, v_{n,h})(P) = \nabla_x \hat{f}(u_{n,h}(P), \varepsilon_n) \cdot v_{n,h}(P)$ for $\forall P \in \Sigma_h$.

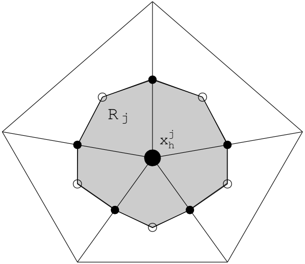
and

$$\langle \varphi_n(u_{n,h}), v_{n,h} \rangle = \int_\Omega \nabla_x \hat{f}(u_{n,h}, \varepsilon_n) \cdot v_{n,h}\, dx, \quad \forall u_{n,h}, v_{n,h} \in V^1_{n,h} := V^1_h$$

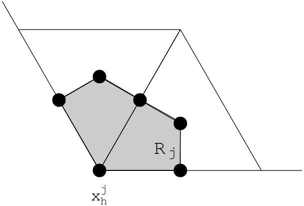
$$
\begin{aligned}
\langle \varphi^{(1)}_{n,h}(u_{n,h}), v_{n,h} \rangle &= \int_\Omega w_{n,h}(u_{n,h}, v_{n,h})\, dx \\
&= \sum_s \int_{T_s} w_{n,h}(u_{n,h}, v_{n,h})\, dx \\
&= \sum_s \frac{\text{meas}\,(T_s)}{3} \big[\nabla_x \hat{f}(u_{n,h}(P_{1s}), \varepsilon_n) \cdot v_{n,h}(P_{1s}) \\
&+ \nabla_x \hat{f}(u_{n,h}(P_{2s}), \varepsilon_n) \cdot v_{n,h}(P_{2s}) \\
&+ \nabla_x \hat{f}(u_{n,h}(P_{3s}), \varepsilon_n) \cdot v_{n,h}(P_{3s}) \big]
\end{aligned}
$$

Consider another partition $\mathcal{R}_h$ of Ω:

$$x_h^j = \{\text{all nodes of the triangulation } \mathcal{T}_h\}$$



x_h^j in the interior



x_h^j at the boundary

$$R_j = conv\,\{\text{mass centres of adjacent triangles, midpoints of adjacent sides }\}$$

$Y_h =$ the space of all piecewise constant functions over $\mathcal{R}_h$:

$$Y_h := \{\mu_h \in L^\infty(\Omega) \,|\, \mu_h|_K \in P_0(R), \forall R \in \mathcal{R}_h\}$$

and the piecewise constant Lagrange interpolation $P_h : W_h^1 \to Y_h$ of w_h is defined as

$$P_h w_h = \sum_j w_h(x_h^j)\chi_{\mathrm{int}_\Omega K_j}(x).$$

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Then

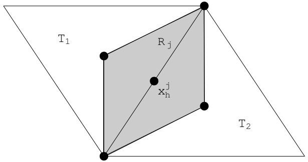
$$\int_\Omega w_{n,h}(u_{n,h}, v_{n,h})\, dx = \int_\Omega \nabla_x \hat{f}(P_h^* u_{n,h}, \varepsilon_n) \cdot P_h^* v_{n,h}\, dx,$$

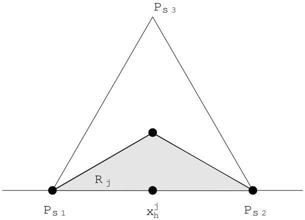
where $P_h^* : V_h^1 \to (Y_h)^m, \quad P_h^* v_h := (P_h v_h^1, \ldots, P_h v_h^m)$.

- **Piecewise quadratic approximations**

Now make the partition $\mathcal{R}_h$ out of quadrilaterals and triangles:

$$x_h^j = \{\text{all midpoints of the sides of the triangles of } \mathcal{T}_h\}$$





for a given midpoint $x_h^j \Rightarrow$

$$R_j = conv\,\{\text{neighbouring nodes, neighbouring mass centres }\}$$

Applications to Mechanics

$\star$ **Unilateral contact with friction**

$\Omega \subset \mathbb{R}^2$ linear elastic body, $\Gamma = \overline{\Gamma}_U \cup \overline{\Gamma}_F \cup \overline{\Gamma}_C$

on Γ_C:

- if $u_N < 0 \Rightarrow S_N = 0,\ S_T = 0$
- if $u_N = 0 \Rightarrow S_N \leq 0$ and

$$-S_T \in \partial j_T(u_T),\ S_T(x) \in \mathbb{R},\ u_T(x) \in \mathbb{R},\ x \in \Gamma_C$$

$$a(u,v) = \int_\Omega C_{ijhk}\varepsilon_{ij}(u)\varepsilon_{hk}(v)\,dx,$$

$$\langle g, v\rangle = \int_\Omega f_i v_i\,dx + \int_{\Gamma_F} F_i v_i\,ds$$

find $u \in K$ such that

$$a(u, v-u) + \int_{\Gamma_C} j_T^0(u_T; v_T - u_T)\, ds \geq \langle g, v-u \rangle, \quad \forall\, v \in K$$

on the function space

$$V = \{v \in H^1(\Omega; \mathbb{R}^2) \,:\, v = 0 \text{ on } \Gamma_U\}$$

and the convex closed cone

$$K = \{v \in V \,:\, v_N \leq 0 \text{ on } \Gamma_C\}.$$

$j_T : \Gamma_C \times \mathbb{R}^2 \to \mathbb{R}$

(i) for all $\eta \in \mathbb{R}$ the function $j_T(\cdot, \eta)$ is measurable on Γ_C;

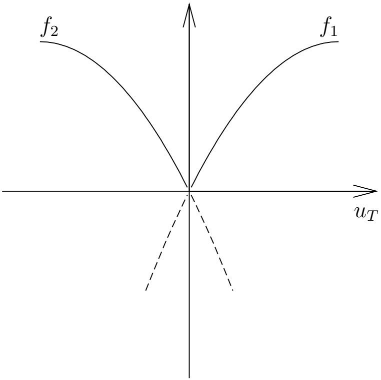
(ii) for almost all $\xi \in \Gamma_C$, the function $j_T(\xi, \cdot)$ is locally Lipschitz;

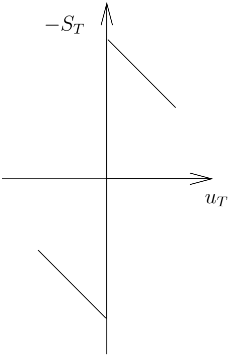
(iii) for almost all $\xi \in \Gamma_C$ and each $\eta \in \mathbb{R}^2$

$$\eta^* \in \partial j_T(\xi, \eta) \Rightarrow |\eta^*| \leq c(1 + |\eta|)$$

$$\eta^* \cdot (-\eta) \leq b(1 + |\eta|)$$

for some constants $c \geq 0$ and $b \geq 0$ not depending on $\xi \in \Gamma_C$.





$$j_T(u_T) = \max\{f_1(u_{T_1}, u_{T_2}), f_2(u_{T_1}, u_{T_2})\}$$

$$-S_T \in \partial j_T(u_T)$$

$$\mathcal{N}_h = \{P \in \Sigma_h \,:\, P \in \overline{\Gamma}_C\}$$

Appoximate V by

$$V_h^1 = \{v_h \in C(\overline{\Omega}; \mathbb{R}^2) \cap V \,:\, v_{h|_T} \in (P_1)^2, \forall T \in \mathcal{T}_h, \, v_{h|_{\Gamma_U}} = 0\}$$

and K by

$$K_h = \{v_h \in V_h^1 \,:\, (v_h)_N(P) \leq 0, \forall P \in \mathcal{N}_h\}.$$

Then W_h^1 is defined as

$$W_h^1 = \{w_h \in C(\overline{\Gamma}_C) \,:\, w_{h|_\Delta} \in P_1, \forall \Delta \in \mathcal{T}_h^C, \},$$

where $\mathcal{T}_h^C$ is the partition of Γ_C induced by $\mathcal{T}_h$.

⋆ Adhesive contact problem

$j_N : \mathbb{R} \to \mathbb{R}$ locally Lipschitz function;

$$-S_N \in \partial j_N(u_N),\ \ S_T = C_T(x) \quad \text{given on } \Gamma_C.$$

Find $u \in K$ such that

$$\langle Au - g, v - u\rangle_V + \int_{\Gamma_C} j_N^0(u_N; v_N - u_N)\, ds \geq 0, \quad \forall v \in K,$$

where

$$\langle g, v\rangle = \int_\Omega f_i v_i\, dx + \int_{\Gamma_C} C_{T_i} v_{T_i} ds,$$

$$V = \{v \in H^1(\Omega; \mathbb{R}^2)\ :\ v = 0 \text{ on } \Gamma_U\}$$

and $K = \{v \in V\ :\ v_N \leq 0 \text{ on } \Gamma_C\}$.

⋆ Nonmonotone skin friction problem

Let Ω be a polygonal domain occupied by an elastic plate or by an elastic body within planar linear elasticity. We split the body forces F in two parts $F_i = \bar{F}_i + \bar{\bar{F}}_i$, $i = 1, 2$, where $\bar{\bar{F}}_i \in L^2(\Omega)$ are given. On a polygonal subdomain $\Omega_0 \subset \Omega$ we consider the multivalued reaction-displacement law

$$-\bar{F}(x) \in \partial j(u(x)) \quad \text{a.e.} \;\; x \in \Omega_0.$$

The subspace V_0 is determined by the Dirichlet bc on Γ_U.
HVI: Find $u \in V_0$ such that

$$a(u, v-u) + \int_{\Omega_0} j^0(u; v-u)\, dx \geq \langle l, v-u \rangle, \quad \forall v \in V_0,$$

where $l \in V_0^*$ is defined as

$$\langle l, v \rangle = \int_{\Gamma_F} F_i v_i \, d\Gamma + \int_{\Omega} \bar{\bar{F}}_i v_i \, dx$$

and $\Gamma = \overline{\Gamma_U} + \overline{\Gamma_F}$.