Summer School, Lecture Notes

Minimax Programming

Part I. Real Variables

Hang-Chin Lai

Chung-Yuan Christian University

Chung Li, Taiwan.

GCM9, July 15-19, 2008

1 Min-Max Theorems

1.1 Introduction to minimax problem by a two person game.

Let X : a normed space, X^* : its normed dual.

$A \subset X$, $B \subset X^*$ fixed subsets.	The special case: $X = \mathbb{R}^n,\ X^* = (\mathbb{R}^n)^* = \mathbb{R}^n$. In this case, X is a reflexive Banach space.

Consider two players A and B;

player A selects a vector x from his strategy A

player B selects a vector x^* from his strategy B

The quantity $\langle x, x^* \rangle$ is computed and player A pays that amount to player B.

Thus A seeks to make his selection to minimize $\langle x, x^* \rangle$, and

B seeks to maximize $\langle x, x^* \rangle$.

If the quantities

$$\overline{\mu} = \min_{x \in A} \max_{x^* \in B} \langle x, x^* \rangle$$

$$\underline{\mu} = \max_{x^* \in B} \min_{x \in A} \langle x, x^* \rangle$$

exist, then player A will find a best choice x_0 so that

A **loses no more than** $\max_{x^* \in B} \langle x_0, x^* \rangle$.

On the other hand, player B will find x_0^* that

B **wins at least** $\min_{x \in A} \langle x, x_0^* \rangle$.

Therefore by their proper choices x_0 and x_0^*, it would have that

$$\underline{\mu} \leq \langle x_0, x_0^* \rangle \leq \overline{\mu}, \qquad \text{(Saddle point property) as well as}$$

$$\underline{\mu} = \max_{x^* \in B} \langle x_0, x^* \rangle \leq \langle x_0, x_0^* \rangle \leq \min_{x \in A} \langle x, x_0^* \rangle = \overline{\mu}.$$

Question arises that whether

$$\underline{\mu} = \overline{\mu}$$

so that the existence of a unique **pay-off** made for **optimal play** by both players.

We state the min-max theorem for above type game (cf. The Book: D.G. Luenberger: Optimization by vector space methods, 1969)

Theorem (Min-Max)

Let X be reflexive normed space, and let $A \subset X$ and $B \subset X^*$ be compact, convex subsets respectively. Then

$$\min_{x \in A} \max_{x^* \in B} \langle x, x^* \rangle = \max_{x^* \in B} \min_{x \in A} \langle x, x^* \rangle.$$

Proof. Let

$$f(x) = \max_{x^* \in B} \langle x, x^* \rangle, \quad g(x) = 0 \quad \text{for all } \; x \in A,$$

one can deduce the conjugate functions as

$$f^*(x^*) = 0, \quad g^*(x^*) = \max_{x \in A} \langle x, x^* \rangle \quad \text{for } x^* \in B.$$

Then applying **Fenchel duality theorem** to yield

$$\begin{aligned} \min\nolimits_{x \in A} f(x) &= \min_{x \in A} \Big\{ f(x) - g(x) \Big\} \\ &= \max_{x^* \in B} \Big\{ g^*(x^*) - f^*(x^*) \Big\} \\ &= \max_{x^* \in B} \min_{x \in A} \langle x, x^* \rangle. \end{aligned}$$

Here $f^*(x^*)$ is the conjugate function of $f(x)$,

$$f^*(x^*) = \sup \Big\{ \langle x, x^* \rangle - f(x) \Big\}.$$

The amount $\Big\{ \langle x, x^* \rangle - f(x) \Big\}$ is finite only if $x^* \in B$. Thus Fenchel duality theorem is applicable. □

1.2 General Minimax Problems

There are various general version of Min-Max theorems. We state several minimax theorems for general functional with arbitrary sets X and Y, $f : X \times Y \to \mathbb{R}$ as the following problems.

(See Ky Fan [1], [2])

[1] Ky Fan, *Minimax Theorems*, Proc. N.A.S. Vol.39(1953), 42-47.

[2] Ky Fan, *Fixed-Point and Minimax Theorems in Locally Convex Topological Linear Spaces*, Proc. N.A.S. Vol.38(1952), 121-126.

Question: Let X and Y be **arbitrary sets** (not necessary topologied).

$f : X \times Y \to \mathbb{R}$. Then, weather $f(x, y)$ has property

$$(\star) \qquad \min_{x \in X} \max_{y \in Y} f(x, y) = \max_{y \in Y} \min_{x \in X} f(x, y).$$

Theorem 1.

X, Y: Compact Hausdorff spaces,

$$f : X \times Y \to \mathbb{R} \quad \begin{cases} l.s.c. \text{ on } X \\ u.s.c. \text{ on } Y \end{cases}. \text{ Then}$$

$$(\star) \text{ holds} \iff \begin{cases} \forall \{x_1, \cdots, x_n\} \subset X \text{ and } \forall \{y_1, \cdots, y_m\} \subset Y, \\ \exists (x_0, y_0) \in X \times Y \text{ (called \textbf{saddle point}), such that} \\ f(x_0, y_k) \le f(x_0, y_0) \le f(x_i, y_0), \ \ 1 \le i \le n, 1 \le k \le m. \end{cases}$$

In particular,

if $f(x, y)$ is convex / concave on X / Y, then $(\star)$ holds.

Theorem 2.

X: Compact Hausdorff space, (not linear sp.)

Y: arbitrary set, (no topology)

$f : X \times Y \to \mathbb{R}$ is l.s.c. on X.

If (i) $x \longrightarrow f(x, y)$ is **convex-like** on X for $y \in Y$,

(ii) $y \longrightarrow f(x, y)$ is **concave-like** on Y for $x \in X$,

then $(\star)$ holds.

Definition.

$f : X \times Y \to \mathbb{R}$ is **convex-like / concave-like** on X / Y, resp

If $\forall x_1,\ x_2 \in X$ and $\zeta_1,\ \zeta_2 \geq 0$ with $\zeta_1 + \zeta_2 = 1$

$\Big(\ \forall y_1,\ y_2 \in Y$ and $\eta_1,\ \eta_2 \geq 0$ with $\eta_1 + \eta_2 = 1\ \Big)$,

then $\exists\, x_0 = x_0(x_1, x_2)$ / $y_0 = y_0(y_1, y_2)$

such that $\forall y \in Y\ \big(\forall x \in X\big)$,

$$f(x_0, y) \leq \zeta_1 f(x_1, y) + \zeta_2 f(x_2, y)$$

$$\Big(\ f(x, y_0) \geq \eta_1 f(x, y_1) + \eta_2 f(x, y_2)\ \Big).$$

Note that: X and Y are **not** necessary **linear spaces**.

Theorem 3.

Let X, Y: arbitrary sets. If $f : X \times Y \to \mathbb{R}$ is almost periodic function, then

$$(\star)\ \textit{holds} \iff \begin{cases} \forall \varepsilon > 0,\ \forall \{x_1, \cdots, x_n\} \subset X,\ \forall \{y_1, \cdots, y_m\} \subset Y, \\ \exists (x_0, y_0) \in X \times Y \text{ called } \text{"}\varepsilon\text{-}\textbf{saddle point}\text{", such that} \\ f(x_0, y_k) - f(x_i, y_0) \le \varepsilon,\ \ 1 \le i \le n, 1 \le k \le m. \\ \text{equivalently,} \\ \quad f(x_0, y_k) \le f(x_0, y_0) + \varepsilon \le f(x_i, y_0) \end{cases}$$

In particular,

if f is convex-like / concave-like on X / Y, respectively, then $(\star)$ holds.

Definition.

$f : X \times Y \to \mathbb{R}$ is **almost periodic**

if it is **left** / **right almost periodic** on X / Y.

That is, if $f(x, y)$ is bounded and $\forall \varepsilon > 0$,

$\exists$ a finite covering $\bigcup_{i=1}^{n} X_i$ (left) $\left(\bigcup_{j=1}^{m} Y_j \text{ (right)} \right)$

s.t. $|f(x', y) - f(x'', y)| < \varepsilon,\ \forall y \in Y$

$\left(|f(x, y') - f(x, y'')| < \varepsilon,\ \forall x \in X \right)$

whenever $x',\ x''$ belong to the same X_i

$\left(\text{whenever } y',\ y'' \text{ belong to the same } Y_j \right)$

Problem. If $f(x, y)$ is a fractional function:

$$f(x, y) = \frac{\varphi(x, y)}{\psi(x, y)}, \quad (x, y) \in X \times Y,$$

then whether the minimax theorem holds?

Note that

(1) convexity or concavity of φ and ψ

$$\not\Rightarrow \quad \text{convexity or concavity of } f = \frac{\varphi}{\psi}.$$

(2) l.s.c. / u.s.c. of φ and ψ

$$\not\Rightarrow \quad \text{l.s.c. / u.s.c. for } f = \frac{\varphi}{\psi}.$$

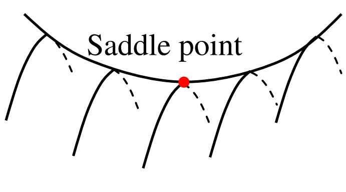
Remark.

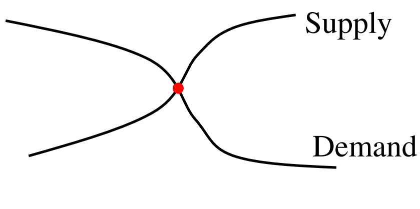
Minimax problem in **Economics** is often related to

Minimum cost / Maximum profit.

Saddle point property is the equilibrium point related to

Supply / Demand.





2 Optimization Problems and Lagrange Multipliers

Consider a general convex programming problem as the form

$$\text{(P)} \qquad \inf \ f(x)$$

$$\text{s.t. } x \in \Gamma_P = \left\{x \in \Omega \in X \mid g(x) \leq_C \mathbf{0}\right\},$$

where $f : X \to \mathbb{R}$ and $g : X \to (Y, C)$ are convex in the convex subset $\Omega \in X,\ C \in Y$ is a pointed convex closed cone (= positive cone Y_+), Y is a normed space with order cone C.

Let $C^* \subset Y^*$ be the dual cone of C. That is , for $y^* \in C^*,\ y^*(y) \geq 0$ for any $y \in C$. The **Lagrange functional** for (P) is defined by

$$L(x, y^*) = f(x) + \langle g(x), y^* \rangle,\ (x, y^*) \in X \times Y^*,$$

and $y^* \in Y^*$ is called **Lagrange multipliers**.

Theorem 1. (Necessary Conds.)

Let $x_0 \in \Gamma_P$ be a (P)-optimal. Suppose that $\overset{\circ}{C} \neq \emptyset$ and there exists a point $x_1 \in \Omega$ such that $g(x_1) <_C \mathbf{0}$.

Then there exists a $y_0^* \in C^*$ such that

$$f(x_0) = \inf\left\{f(x) + \langle g(x), y_0^* \rangle\right\},$$

and $$\langle g(x_0), y_0^* \rangle = 0.$$

That is, (x_0, y_0^*) is a **saddle point** of the Lagrangian

$$L(x, y^*) = f(x) + \langle g(x), y^* \rangle, \quad (x, y^*) \in X \times Y^*.$$

Theorem 2. (Sufficient Conds.)

In problem (P), if the Lagrangian

$$L(x, y^*) = f(x) + \langle g(x), y^* \rangle,$$

possesses a **saddle point** $(x_0, y_0^*) \in \Omega \times C^*$, then x_0 solves the problem (P). That is

$$f(x_0) = \inf_{x \in \Gamma_P} f(x).$$

The **Lagrange dual functional** w.r.t. (P) is given by

$$\varphi(y^*) = \inf \left\{ f(x) + \langle g(x), y^* \rangle \right\}, \;\; y^* \in C^*.$$

In general φ may not be finite throughout the positive cone $C^* \subset Y^*$, but in the region which φ takes finite value is concave and represented by

$$\varphi(y^*) = \inf \left\{ w(y) + \langle y, y^* \rangle \right\}$$

where $$w(y) = \left\{ \inf \; f(x) \mid x \in \Omega, g(x) \leq_C y \quad \in Y \right\}.$$

The problem

$$(\mathrm{P}^*) \qquad \max_{y^*\in C^*} \varphi(y^*) = \sup_{y^*\in C^*} \inf_{x\in\Gamma_P} \left\{ f(x) + \langle g(x), y^* \rangle \right\}$$

is called **Lagrange dual** with respect to problem (P).

Then we have the **Lagrange Duality Theorem:**

Theorem 3. (Lagrange Duality)

Let (P) be a convex programming problem:

$$(P) \qquad \inf_{x\in\Gamma_P} f(x)$$

$$\text{s.t.} \quad \Gamma_P = \left\{ x \in \Omega \mid g(x) \leq_C \mathbf{0}, \quad \overset{\circ}{C} \neq \emptyset \right\}.$$

Then

$$(\star) \qquad \inf_{x\in\Gamma_P} f(x) = \max_{y^*\in C^*} \varphi(y^*)$$

where maximum on the right is achieved for some $y_0^* \geq 0$ in Y^*.

Corollary. Furtheremore in Theorem 3,

if on the left of $(\star)$ is achieved at some $x_0 \in \Omega$, then

$$\langle g(x_0), y_0^* \rangle = 0$$

and x_0 minimizes $\left\{ f(x) + \langle g(x), y_0^* \rangle, \ x \in \Gamma_P \right\}$.

That is the minimax theorem of $L(x, y^*)$ holds:

$$\inf_{x\in\Gamma_P} \sup_{y^*\in C^*} L(x, y^*) = \sup_{y^*\in C^*} \inf_{x\in\Gamma_P} L(x, y^*) = L(x_0, y_0^*)$$

The above results hold for

$$X = \mathbb{R}^n,\ Y = \mathbb{R}^m,\ C = \mathbb{R}^m_+.$$

In this case, X and Y are reflexive Banach spaces, and the Lagrange multiplier $y^* \in C^*$ is usually denoted by

$$y^* = (\lambda_1, \cdots, \lambda_m) \in (\mathbb{R}^m_+)^* = \mathbb{R}^m_+,\ \lambda_k \geq 0\ (k = 1, \cdots, m).$$

3 Minimax Programming Problems

Now we turn to introduce the minimax programming problems including real variables as well as their duality theory.

In general a minimax programming problem is considered as the form:

$$\text{(P)} \quad \min_{x \in X} \max_{y \in Y} f(x, y)$$

$$\text{s.t.} \quad h(x) \leq_{\mathbb{R}^r_+} \mathbf{0}.$$

In 1977 Scmittendorff: *"Necessary conditions and sufficient conditions for static minimax problems"*, JMAA. 57, pp683-693,

was firstly studied the problem (P), where Y is a compact subset of $\mathbb{R}^m$, $X = \mathbb{R}^n$, $f(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $h(\cdot) : \mathbb{R}^n \to \mathbb{R}^p$. are C^1**-mappings**.

Usually, in the study of problem (P), the main purpose is to establish the necessary / sufficient optimality conditions.

If the necessary optimality conditions is established as well, the **sufficient optimality conditions** can be regarded as the **converse of necessary conditions with extra assumptions**.

Thus the sufficient theorems are various. Many authors effort to search such **extra conditions**. For example, convexity as well as the variety of generalized convexity are explored.

Based on the optimality conditions, the duality theory corresponding to the primal problem are developed.

The following references are related to minimax fractional programming in the recent years. (cf. [1~6])

In [1], Lai et al. investigated $f(x, y)$ in (P) by a nondifferentiable fractional minimax problem with objective function:

$$f(x, y) = \frac{f_i(x)}{g_i(x)} = f(x, i), \quad y = i \in Y = \{1, \cdots, p\}.$$

Precisely problem (P) becomes

$$(P_I) \qquad \min_{x\in\Gamma_P} \max_{1\le i\le p} \frac{f_i(x)}{g_i(x)}$$

$$\text{s.t. } \Gamma_P = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{l} h_j(x) \le 0,\ j = 1, \cdots, r \\ \text{with } h_j(\widetilde{x}) < 0 \text{ for some } \widetilde{x} \end{array} \right\}$$

where f_i, $-g_i$ and h_j are continuous real valued convex functions and $g_i > 0$ on Γ_P for all i. The paper established the necessary and sufficient optimality theorem as the following form:

Theorem.

The point $x_0 \in \Gamma_P$ is (P_I)-optimal.

$\Longleftrightarrow$ $\exists\, \varphi(x_0) \in \mathbb{R}_+$, Lagrange multiplier,

$\alpha_\ell \ge 0,\ \ell \in I = \{1, \cdots, k\}$ with $\sum_{\ell\in I} \alpha_\ell = 1$

and $\mu_j \ge 0$ as $h_j(\widetilde{x}) < 0$ for some $\widetilde{x} \in \Gamma_P$, such that

$$0 \in \sum_{\ell\in I} \alpha_\ell \partial f_i(x_0) + \varphi(x_0)\big(\partial(-g_i(x_0))\big) + \sum_{j=1}^{r} \mu_j \partial h_j(x_0)$$

where I is the set of all indexes for which

$$f_\ell(x_0) - \varphi(x_0) g_\ell(x_0) = \max_{1\le i\le p} \big(f_i(x_0) - \varphi(x_0) g_i(x_0)\big),$$

and $\varphi(x_0) = \lambda^*$ is the optimal value of (P).

Remark. In (P_1), since f_i, $-g_i$ and h_j are continuous convex functions, thus each function is subdifferentiable. If x_0 is (P_1)-optimal with value

$$\lambda^* = \max_{1\le i\le p} \frac{f_i(x_0)}{g_i(x_0)} \equiv \varphi(x_0),$$

and

$$f_\ell(x_0) - \varphi(x_0)g_\ell(x_0) = \max_{1\le i\le p}(f_i(x_0) - \varphi(x_0)g_i(x_0)),$$

then

$$\varphi(x_0) = \inf_{x\in\Gamma_P} \max_{1\le i\le p} \frac{f_i(x_0)}{g_i(x_0)}.$$

If $Y \subset \mathbb{R}^m$ is a nondiscrete compact subset, we can also consider the following **nondifferentiable** minimax fractional programming problem:

$$(P_2) \qquad \min\ F(x) = \sup_{y\in Y} \frac{f(x,y) + (x^TAx)^{1/2}}{g(x,y) - (x^TBx)^{1/2}}$$

$$\text{s.t.} \quad x \in \mathbb{R}^n \text{ and } h(x) \leq_{\mathbb{R}^p_+} \mathbf{0}$$

where $Y \subset \mathbb{R}^m$ is a compact subset, $f(\cdot,\cdot)$ and $g(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ are C^1 functions and $h(\cdot) : \mathbb{R}^n \to \mathbb{R}^p$ is a C^1 mapping; A and B are $n \times n$ positive semidefinite matrices. Then the necessary / sufficient optimality conditions was established in Ref [2].

We remark here in the fractional function, the term $(x^TAx)^{1/2}$ as well as $(x^TBx)^{1/2}$ are **not differentiable at the point** x_0 where either $x_0^TAx_0 = 0$ or $x_0^TBx_0 = 0$. Thus the fractional function is not differentiable even f and g are C^1 functions. Therefore the sufficient conditions of (P_2) is not invertible from necessary conditions, and we need to assume the some generalized convexity as the extra assumptions to the necessary conditions. See Ref [2] for detail.

This lecture will be stop here. Next lecture, we would like to introduce some minimax programming (fractional or nonfractional) problems for **set variables** as well as **complex variables**.

References (for Real Variables)

[1] Lai/Liu/Tanaka,"*Duality without a constraint qualification for minimax fractional programming*", JOTA 101(1) (1999) pp109-125.

[2] Lai/Liu/Tanaka,"*Necessary and sufficient conditions for minimax fractional programming*", JMAA 230 (1999) pp311-328.

[3] Lai/Lee,"*On duality theorems for a nondifferentiable minimax fractional programming*", J. CAM 146 (2002) pp115-126.

[4] Lai/Liu,"*Minimax fractional programming involving generalized invex functions*", The ANZIAM J. 44 (2003) pp339-354.

[5] Lai, H.C.,"*On a dynamic fractional game*", JMAA 294(2) (2004) pp644-654.

[6] Lai, H.C.,"*Sufficiency and duality of fractional programming with generalized invexity*", Taiwanese J. Math. 10(6) (2006) pp1685-1695.