Summer School, Lecture Notes

Minimax Programming

Part II. Set Variables / Complex Variables

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1 Introduction

In Part I, the minimax programming with real variables is introduced. Now we will take **set variables** or **complex variables** in a programming problem. That is, functions in programming problems are taken by **set functions** or **complex variable functions**. At first we introduce the analysis of set functions in optimization theory. The concept is firstly developed by

Morris: *Optimal constrained selection of measurable subsets*,

JMAA 70(1979), 546-562.

After then, many researchers have further development in their own interesting including theory and applications. Throughout this lecture, let (X, Γ, μ) be a **finite atomless** measure space with $L_1(X, \Gamma, \mu)$ separable. That is, $\mu(X) < \infty$, and for any measurable set $A \in \Gamma$ with $\mu(A) > 0$, $\exists$ a nonempty set $B \subset A$, s.t. $\mu(B) > 0$. Thus, each $\Omega \in \Gamma$ corresponds characteristic function $\chi_\Omega \in L_\infty = L_1^* \subset L_1$, and so for $f \in L_1$, and $\chi_\Omega \in L_\infty$, the dual pair $\langle f, \chi_\Omega \rangle$ is represented by

$$\int_{\Omega} f d\mu = \langle f, \chi_{\Omega} \rangle.$$

By the separability of $L_1 \supset L_\infty$, all topologies induced in Γ is topology induced by w^*-topology on $\{ \chi_\Omega \mid \Omega \in \Gamma \} \subset L_\infty$, and the family $\{ \chi_\Omega \mid \Omega \in \Gamma \}$ can be taken countable sequences which are dense in L_1.

The convexity, differentiability, subdifferentiability can be defined for set functions. Readers are encouraged to consult Morris' paper and

Chen/Lai: *"Optimization Analysis Involving Set Functions"*,

Applied Math E-Notes 2(2002), 78-97.

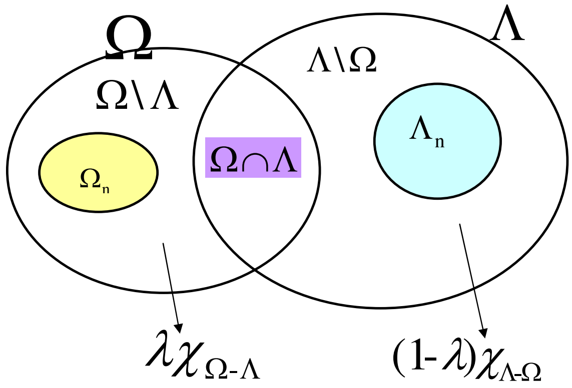
This talk is focus on the minimax programming problems involving set variables / complex variables which we have developed in recent 10 years (cf. Lai et al. [1]~[12] and the references therein).

(A) Set Variable Case

Let (X, Γ, μ) be atomless finite measure space with $L^1(X, \Gamma, \mu)$ separable. For any $(\Omega, \Lambda, \lambda) \in \Gamma \times \Gamma \times [0, 1]$, and any sequences $\{\Omega_n\}$ in

Ω and $\{\Lambda_n\}$ in Λ such that

$$\chi_{\Omega_n} \xrightarrow{w^*} \lambda \chi_{\Omega \backslash \Lambda} \quad \text{and} \quad \chi_{\Lambda_n} \xrightarrow{w^*} (1-\lambda)\chi_{\Lambda \backslash \Omega} \qquad (\star)$$
$$\Longrightarrow \quad \chi_{\Omega_n \cup \Lambda_n \cup (\Omega \cap \Lambda)} \xrightarrow{w^*} \lambda \chi_{\Omega} + (1-\lambda)\chi_{\Lambda}, (\text{ as } n \to \infty).$$



Let $M_n = \Omega_n \cup \Lambda_n \cup (\Omega \cap \Lambda)$ satisfy $(\star)$, namely **Morris sequence**.

A subfamily $\mathcal{S} \subset \Gamma$ is said to be **convex family** if $\mathcal{S}$ possesses the property $(\star)$ for any given $(\Omega, \Lambda, \lambda) \in \mathcal{S} \times \mathcal{S} \times [0,1]$.

Definition 1. A set function $F : \Gamma \to \mathbb{R}$ is called **convex** on a convex family $\mathcal{S}$ in Γ if for any $(\Omega, \Lambda, \lambda) \in \mathcal{S} \times \mathcal{S} \times [0,1]$, $\exists$ a Morris sequence $\{M_n\}$ in Γ such that

$$\overline{\lim_{n\to\infty}} F(M_n) \leq \lambda F(\Omega) + (1-\lambda)F(\Lambda).$$

Definition 2. A set function $F : \Gamma \to \mathbb{R}$ is **differentiable**

at $\Omega_0 \in \Gamma$, if $\exists\, f_{\Omega_0} \in L_1$, namely the derivative of F at Ω_0,

$$\left(f_{\Omega_0} \equiv F'(\Omega_0)\right)$$

s.t.

$$F(\Omega) = F(\Omega_0) + \langle\, f_{\Omega_0}, \chi_\Omega - \chi_{\Omega_0} \,\rangle + o\Big(\rho(\Omega, \Omega_0)\Big)$$

where ρ is a **pseudometric** on Γ, defined by

$$\rho(\Omega_1, \Omega_2) = \mu(\Omega_1 \Delta \Omega_2), \quad \Omega_1, \Omega_2 \in \Gamma$$

where Δ stands for the symmetric difference of sets in Γ.

Definition 3. An element $f \in L_1$ is a **subgradient** of a set

function F at $\Omega_0 \in \Gamma$, if it satisfies the inequality

$$F(\Omega) \geq F(\Omega_0) + \langle\, f, \chi_\Omega - \chi_{\Omega_0} \,\rangle \quad \text{for any } \Omega \in \Gamma.$$

The set of all subgradients of F at Ω_0 is called the

subdifferential of F at Ω_0, and is denoted by

$$\partial F(\Omega_0) \equiv \Big\{ f \in L_1 \;\Big|\; F(\Omega) \geq F(\Omega_0) + \langle\, f, \chi_\Omega - \chi_{\Omega_0} \,\rangle,\ \Omega \in \Gamma \Big\}.$$

Remark. $F : \Gamma \to \mathbb{R}$ is **subdifferentiable** at Ω_0 if

$$\partial F(\Omega_0) \neq \emptyset.$$

A convex set function is subdifferentiable.

Definition 4. The **conjugate functional** F^* of F is defined by

$$F^*(f) = \sup_{\Omega \in \Gamma} \left\{ \langle f, \chi_\Omega \rangle - F(\Omega),\ f \in L_1 \right\},\ F \not\equiv \infty.$$

The **biconjugate function** F^{**} of F is defined by

$$\begin{aligned} F^{**}(\Omega) &= \sup_{f \in L_1} \left\{ \langle f, \chi_\Omega \rangle - F^*(f) \right\} \quad \text{for } \Omega \in \text{Dom } f \\ &= \quad \infty \quad \text{if } \Omega \notin \text{Dom } f. \end{aligned}$$

Theorem 1. (Fenchel Moreaus)

Let F be proper convex w^*-l.s.c. set function on its convex domain $\mathcal{S}$. Then

$$F^{**}(\Omega) = F(\Omega), \quad \forall\ \Omega \in \Gamma.$$

One find a nondifferentiable set function in minimax fractional programming problem as follows

$\Big($ Similar to Part I (p.14) §3 , Problem $(\text{P}_1)\Big)$

$$(\text{P}) \qquad \min \max_{1 \le i \le p} \frac{F_i(\Omega)}{G_i(\Omega)}$$

$$\text{s.t. } \Omega \in \mathcal{S} \subset \Gamma \text{ and}$$

$$H_j(\Omega) \le 0,\ j \in \{1, \cdots, m\}$$

where $\mathcal{S}$ is a convex subfamily of Γ; F_i, $-G_i$, $(i = 1, \cdots, p)$ and

H_j, $(j = 1, \cdots, m)$ are **convex set functions**, and given some natural assumptions in (P), for instance, $G_i(\cdot) > 0$ and $F_i(\cdot) \geq 0$, $H_j(\Omega) < 0$ for some $\Omega \in \mathcal{S}$, and all convexities set function are proper convex and w^*-continuous etc.

Problem (P) $\Big($ see Lai [1]$\Big)$ can be reduced to an **equivalent non-fractional** parametric problem as the form:

$$\begin{aligned} \text{(EP)} \quad & \text{Minimize } \lambda \\ & \text{s.t.} \quad F_i(\Omega) - \lambda G_i(\Omega) \leq 0, \ \ 1 \leq i \leq p, \\ & \qquad\ \ H_j(\Omega) \leq 0, \ \ 1 \leq j \leq m, \\ & \qquad\ \ \Omega \in \mathcal{S}. \end{aligned}$$

$\Big($ cf. Zalmai, Optimaization 20(1989), pp.377-395 $\Big)$

If Ω_0 is an optimal solution of (P), then (Ω_0, λ_0) is an optimal solution of (EP) where

$$\lambda_0 = \max_{1 \leq i \leq p} \frac{F_i(\Omega_0)}{G_i(\Omega_0)}$$

is the optimal value of (EP).

Conversely, if (Ω^0, λ^0) is an optimal solution of (EP), then Ω^0 is an optimal solution of (P) with optimal value λ^0, the following expression is immediate.

$$(0) \qquad \phi(\Omega) \equiv \max_{1\le i\le p} \frac{F_i(\Omega)}{G_i(\Omega)} = \max_{u\in I} \frac{\langle u, F(\Omega)\rangle}{\langle u, G(\Omega)\rangle}$$

where $I = \left\{u \in \mathbb{R}^p_+ \mid \sum_{i=1}^p u_i = 1\right\}$.

Theorem 2. (Nec. Optim. Conds.) Let $\Omega_0 \in \mathcal{S}$ be a (P)-optimal with optimal value λ_0, then $\exists$ multipliers $y^* \in \mathbb{R}^p_+$ and $z^* \in \mathbb{R}^m_+$ such that $(\Omega_0, \lambda_0, y^*, z^*)$ satisfies

(1) $0 \in \partial\langle y^*, F(\cdot)\rangle(\Omega_0) + \lambda_0\partial\langle -y^*, G(\cdot)\rangle(\Omega_0)$

$$+\partial\langle z^*, H(\cdot)\rangle(\Omega_0) + N_{\mathcal{S}}(\Omega_0),$$

(2) $\langle y^*, F(\Omega_0) - \lambda_0 G(\Omega_0)\rangle = 0,$

(3) $\langle z^*, H(\Omega_0)\rangle = 0.$

where $F = (F_1, \cdots, F_p)$, $G = (G_1, \cdots, G_p)$ and $H = (H_1, \cdots, H_m)$; $N_{\mathcal{S}}(\Omega_0)$ denotes the **normal cone** to $\mathcal{S}$ at Ω_0. The optimal point Ω_0 is **regular** if (1) $\sim$ (3) are valid and

$\nexists$ $h \in \partial\langle z^*, H(\cdot)\rangle(\Omega_0)$ or $\eta \in N_{\mathcal{S}}(\Omega_0)$ such that

$$h + \eta = 0.$$

As we have explained before, the **sufficiency** for optimal conditions may be regarded as the inverse of Nec. Conds. with **extra**

assumptions. To find such **extra assumptions**, we define a kind of **generalized** $(\mathcal{F}, \rho, \theta)$**-convexity** as in the following definitions.

Let $\mathcal{F} : \Gamma \times \Gamma \times L_1 \to \mathbb{R}$ be **sublinear functional** on the third argument in L_1, $F : \Gamma \to \mathbb{R}$ a set function. Let $\rho \in \mathbb{R}$ and $\theta : \Gamma \times \Gamma \to \mathbb{R}_+ = [0, \infty)$ such that $\theta(\Omega, \Omega_0) \neq 0$ if $\Omega \neq \Omega_0$.

Then we give the following essential definitions as the explanation for **extra assumptions** added to the Nec. Optim. Conds.

(Refer to Lai / Liu, JAMM 215(1997) pp.443-460 for generalized $(\mathcal{F}, \rho, \theta)$-convex set functions)

Definition 5. (Generalized $(\mathcal{F}, \rho, \theta)$ Convexities)

For each $\Omega \in \Gamma$ and $f \in \partial F(\Omega_0) (\subset L_1)$, we define

1. F is $(\mathcal{F}, \rho, \theta)$**-convex** at Ω_0, if

$$F(\Omega) - F(\Omega_0) \geq \mathcal{F}(\Omega, \Omega_0; f) + \rho\theta(\Omega, \Omega_0)$$

2. F is $(\mathcal{F}, \rho, \theta)$**-quasiconvex** at Ω_0, if

$$F(\Omega) \leq F(\Omega_0) \Longrightarrow \mathcal{F}(\Omega, \Omega_0; f) \leq -\rho\theta(\Omega, \Omega_0)$$

3. F is **prestrictly** $(\mathcal{F}, \rho, \theta)$**-quasiconvex** at Ω_0, if

$$F(\Omega) < F(\Omega_0) \Longrightarrow \mathcal{F}(\Omega, \Omega_0; f) \leq -\rho\theta(\Omega, \Omega_0)$$

4. F is $(\mathcal{F}, \rho, \theta)$**-pseudoconvex** at Ω_0, if

$$\mathcal{F}(\Omega, \Omega_0; f) \geq -\rho\theta(\Omega, \Omega_0) \Longrightarrow F(\Omega) \geq F(\Omega_0)$$

5. F is **strictly** $(\mathcal{F}, \rho, \theta)$**-pseudoconvex** at Ω_0, if

$$\mathcal{F}(\Omega, \Omega_0; f) \geq -\rho\theta(\Omega, \Omega_0) \Longrightarrow F(\Omega) > F(\Omega_0)$$

Remark. If $\rho \geq 0$ in above Definition 5 and take the functional $\mathcal{F} : \Gamma \times \Gamma \times L_1 \to \mathbb{R}$ to be

$$\mathcal{F}(\Omega, \Omega_0; f) = \langle \chi_\Omega - \chi_{\Omega_0}, f \rangle,$$

then $(\mathcal{F}, \rho, \theta)$-convexity of F is deduced that F is convex at Ω_0.

The symbol $\langle y^*, F(\cdot) \rangle \equiv y^{*T} F(\cdot)$ is often used as the **inner product** in Euclidian space $\mathbb{R}^p$. By the above definitions for generalized convexity, the sufficient optimality conditions for (P) is established.

For convenience we denote the set functions $A,\ B,\ C$ by

$$A(\Omega) = y^{*T} G(\Omega_0) y^{*T} F(\Omega) - y^{*T} F(\Omega_0) y^{*T} G(\Omega)$$

$$B(\Omega) = z^{*T} H(\Omega)$$

$$C(\Omega) = A(\Omega) + y^{*T} G(\Omega_0) B(\Omega)$$

Then we have the main theorem (including five sufficient optimality conds.) (cf. Lai / Liu [3])

Theorem 3. (Suff. Optim. Conds.)

Let Ω_0 be a feasible solutions of (P). Assume that there exist $y^* \in I$ and $z^* \in \mathbb{R}^m_+$ which satisfy the (Nec.) condition (1)~(3), and let $\mathcal{F}(\Omega, \Omega_0; -\eta) \geq 0$ for each $\eta \in N_{\mathcal{S}}(\Omega_0)$, and Ω be any feasible solution of (P). Furthermore suppose that any one of the following conditions is valid:

(a) $y^{*T} F$ is $(\mathcal{F}, \rho_1, \theta)$-convex at Ω_0,

$-y^{*T} G$ is $(\mathcal{F}, \rho_2, \theta)$-convex at Ω_0,

$z^{*T} H$ is $(\mathcal{F}, \rho_3, \theta)$-convex at Ω_0,

and $y^{*T} G(\Omega_0)\rho_1 + y^{*T} F(\Omega_0)\rho_2 + z^{*T} H(\Omega_0)\rho_3 \geq 0$.

(b) A is $(\mathcal{F}, \rho_1, \theta)$-pseudoconvex at Ω_0,

B is $(\mathcal{F}, \rho_2, \theta)$-quasiconvex at Ω_0,

and $\rho_1 + y^{*T}G(\Omega_0)\rho_2 \geq 0$.

(c) A is $(\mathcal{F}, \rho_1, \theta)$-quasiconvex at Ω_0,

B is strictly $(\mathcal{F}, \rho_2, \theta)$-pseudoconvex at Ω_0,

and $\rho_1 + y^{*T}G(\Omega_0)\rho_2 \geq 0$.

(d) C is $(\mathcal{F}, \rho, \theta)$-pseudoconvex at Ω_0 and $\rho \geq 0$.

(e) C is prestrictly $(\mathcal{F}, \rho, \theta)$-quasiconvex at Ω_0 and $\rho > 0$.

Then Ω_0 is an optimal solution of (P).

Employing the Suff. Optim. Conds., one can proceed to establish the three duality theorems:

weak, strong, and strict converse theorem

for the **Wolfe type** dual, the **Mond-Weir type** dual and the **parametric type** dual programming problems (see Lai et al, [2-3]).

Furthermore, a **mixed type dual** model is investigated in Lai et al, [4-5].

(B) Complex Variable Case

Mathematical programming in complex space was studied first by Levinson in 1966 for linear programming. Hence after many authors, for example,

Abrams, Craven, Swarup and Shrma, Datta and Bhatia,

Parkash, Ferrero, Lai, Liu, Schaible, Stancu-Minasian etc.

studied linear fractional, nonlinear fractional or nonfractional programming problems in complex spaces.

Complex programming could be applied to **electrical networks** with alternating current with complex variable $z \in \mathbb{C}^n$ representing **currents** or **voltages** for element of network. It is also employed to variant fields in engineering, like blind deconvolution, blind equalization, minimal entropy, maximum kurtosis, optimal receiver etc. (cf. Lai / Lin [7]).

In [10], Chen / Lai / Schaible introduce a generalized

Charnes-Cooper variable transformation to change fractional complex program. into nonfractional program., and proved that the optimal solution of complex fractional program. can be reduced to an optimal solution of the equivalent nonfractional program. and vice versa.

Recently, in [6-12], Lai et al. are investigated minimax programming in complex spaces. Now we consider a new problem for nondifferentiable (nonfractional) minimax programming as follows:

$$\text{(P)} \qquad \min_{\zeta \in X} \sup_{\eta \in Y} \ Re\Big[f(\zeta, \eta) + (z^H A z)^{1/2}\Big]$$

$$\text{s.t.} \quad X = \Big\{\zeta = (z, \bar{z}) \in \mathbb{C}^{2n} \mid -h(\zeta) \in S\Big\}$$

where $Y = \{\eta = (w, \overline{w}) \mid w \in \mathbb{C}^m\}$ is a compact subset in $\mathbb{C}^{2m}$,

$A \in \mathbb{C}^{n \times n}$ is a positive semidefinite Harmitian matrix,

S is a polyhedral cone in $\mathbb{C}^p$,

$f(\cdot, \cdot)$ is continuous, and for each $\eta \in Y$,

$f(\cdot, \eta) : \mathbb{C}^{2n} \to \mathbb{C}$ and $h(\cdot) : \mathbb{C}^{2n} \to \mathbb{C}^p$ are analytic in Q

$Q = \{\zeta = (z, \bar{z}) \mid z \in \mathbb{C}^n\} \subset \mathbb{C}^{2n}$ is a linear manifold over $\mathbb{R}$.

Problem (P) is **nondifferentiable programming** at the optimal point $\zeta_0 = (z_0, \overline{z_0})$ where $z_0^H A z_0 = 0$.

There are three special cases of problem (P) as follows.

(a) If Y vanishes, then (P) is reduced to one complex variable:

(P$_1$) Minimize $Re\left[f(\zeta) + (z^H Az)^{1/2}\right]$

s.t. $\zeta = (z, \overline{z}) \in X = \{\zeta = (z, \overline{z}) \mid -h(\zeta) \in S\}$.

This form is investigated by Mond and Craven.

[cf. J. Math. Oper. and Stat. 6(1975), pp.581-591]

(b) If $A \equiv 0$, (P) becomes a differentiable complex programming

(P$_0$) $\min_{\zeta \in X} \sup_{\eta \in Y} Re f(\zeta, \eta)$.

s.t. $-h(\zeta) \in S$.

[cf. Datta / Bhatia, JMAA 101(1984), pp.1-11]

(c) Problem (P_0) extended the real minimax programming of

(P$_r$) $\min_{x \in X \subset \mathbb{R}^n} \sup_{y \in Y \subset \mathbb{R}^m} f(x, y)$

s.t. $h(x) \leq 0$ in $\mathbb{R}^p$, $f(\cdot, \cdot)$ and $h(\cdot)$ are C^1 functions.

Problem (P$_r$) was firstly investigated by Schmittendorff,

JMAA, 57(1977), pp.683-693.

Symbols / Definitions

S in $\mathbb{C}^p$ is a **polyhedral cone** if $\exists$ a positive integer k, and a matrix $B \in \mathbb{C}^{k\times p}$ s.t. $S = \{\xi \in \mathbb{C}^p \mid Re(B\xi) \geq 0\}$.

$S^* = \left\{\mu \in \mathbb{C}^p \mid Re\langle \xi, \mu\rangle \geq 0 \ \ \forall\, \xi \in S\right\}$: dual cone of S. $\forall \eta \in Y \subset \mathbb{C}^{2m}$, $f(\cdot, \eta) : \mathbb{C}^{2n} \to \mathbb{C}$ and $h(\cdot) : \mathbb{C}^{2n} \to \mathbb{C}^p$ are analytic at $\zeta_0 = (z_0, \overline{z_0}) \in Q$.

In order to have a convex real part for nonlinear analytic function, the complex functions need to define on the linear manifold

$$Q = \{\zeta = (z, \overline{z}) \in \mathbb{C}^{2n} \mid z \in \mathbb{C}^n\}.$$

(cf. Ferrero in JMAA 164(1992), pp.399-416)

Theorem 4. (Nec. Optim. Conds.)

Let $\zeta_0 = (z_0, \overline{z_0}) \in Q$ be (P)-optimal such that $\langle Az_0, z_0\rangle = 0$, and

$$Z_{\widetilde{\eta}}(\zeta_0) = \left\{ \zeta \in \mathbb{C}^{2n} \,\middle|\, -h'_\zeta(\zeta_0)\zeta \in S(-h(\zeta_0)),\ \zeta = (z, \overline{z}) \in Q \text{ and } Re\Big[\sum_{i=1}^{k} \lambda_i f'_\zeta(\zeta_0, \eta_i)\zeta + \langle Az, z\rangle^{1/2}\Big] < 0 \right\} = \emptyset.$$

Then

$$\sum_{i=1}^{k} \lambda_i\Big[\overline{\nabla_z f(\zeta_0, \eta_i)} + \nabla_{\overline{z}} f(\zeta_0, \eta_i) + Au\Big] + \Big(\mu^T\overline{\nabla_z h(\zeta_0)} + \mu^H \nabla_{\overline{z}} h(\zeta_0)\Big) = 0; \tag{1}$$

$$Re\langle\, h(\zeta_0), \mu\,\rangle = 0; \tag{2}$$

$$u^H A u \leq 1; \tag{3}$$

$$(z_0^H A z_0)^{1/2} = Re(z_0^H A u). \tag{4}$$

where $\lambda_i > 0$, $\sum_{i=1}^{k} \lambda_i = 1$ and finite points

$\eta_i \in Y(\zeta_0) = \left\{\eta \in Y \mid Re\, f(\zeta_0, \eta) = \sup_{\nu \in Y} Re\, f(\zeta_0, \nu)\right\}$, for $i = 1, \cdots, k$.

Theorem 5. (Sufficient Optimality Conditions).

Let $\zeta_0 = (z_0, \overline{z_0}) \in Q$ be a feasible solution of (P). Suppose that $\exists$ a positive integer k, and $\eta_i \in Y(\zeta_0)$, $\lambda_i > 0$, $i = 1, \cdots, k$, with $\sum_{i=1}^{k} \lambda_i = 1$, such that for $0 \neq \mu \in S^* \subset \mathbb{C}^p$, $u \in \mathbb{C}^n$ the conditions (1)~(4) are valid either for $\langle Az_0, z_0 \rangle > 0$ or $\langle Az_0, z_0 \rangle = 0$ with $Z_{\widetilde{\eta}}(\zeta_0) = \emptyset$.

Assume further that any one of the following conditions (i)~(iii) holds:

(i) $Re\left[\sum_{i=1}^{k} \lambda_i f(\zeta, \eta_i) + z^H A u\right]$ is pseudoconvex on $\zeta = (z, \overline{z}) \in Q$,

$h(\zeta)$ is quasiconvex on Q w.r.t. the polyhedral cone $S \subset \mathbb{C}^p$;

(ii) $Re\left[\sum_{i=1}^{k} \lambda_i f(\zeta, \eta_i) + z^H A u\right]$ is quasiconvex on $\zeta = (z, \overline{z}) \in Q$

and $h(\zeta)$ is strictly pseudoconvex on Q w.r.t. $S \subset \mathbb{C}^p$;

(iii) $Re\left[\sum_{i=1}^{k} \lambda_i f(\zeta, \eta_i) + z^H A u + \langle h(\zeta), \mu \rangle\right]$ is pseudoconvex on

$\zeta = (z, \overline{z}) \in Q$.

Then $\zeta_0 = (z_0, \overline{z_0})$ is an optimal solution of (P).

By the above optimality conds., we can establish two dual forms:

parametric dual (D1) and parameter free dual (D2)

problems, and prove three duality theorems:

weak, strong and strict converse theorem.

It is also proved that optimal values between the primal problem and dual problems are no duality gap under some additional conditions.

Q: As a plausible problem for a minimax fractional programming in complex spaces as the form:

$$\min_{\zeta \in X} \sup_{\eta \in Y} \frac{Re\,[f(\zeta,\eta) + (z^H A z)^{1/2}]}{Re\,[g(\zeta,\eta) - (z^H B z)^{1/2}]}$$

$$\text{s.t.} \quad X = \left\{\zeta = (z, \bar{z}) \in \mathbb{C}^{2n} \mid -h(\zeta) \in S \subset \mathbb{C}^p\right\}$$

Y is a compact subset of $\mathbb{C}^{2m}$

Then how about the optimality conditions and the duality theory?

References. A. For Set Variables.

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Thank you very much for your attention!!