2nd Summer School on Generalized Convex Analysis

NSYSU, Kaohsiung, Taiwan ROC

Lecture # 5:

Approximate convexity and submonotonicity

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Approximately convex functions

A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be *approximately convex* (Ngai-Luc-Théra, 2000) (resp., *directionally approximately convex* (Daniilidis-Georgiev, 2004, Georgiev, 1997)) *at* $x_0 \in X$, if:

For every $\varepsilon > 0$ (resp., and $d \in S_X$) there exists $\delta > 0$ such that for all $x, y \in B(x_0, \delta) \cap \mathrm{dom}\, f$ (resp., with $x \neq y$ and $(x-y)/\|x-y\| \in B(d, \delta)$) one has:

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon t(1-t)\|x-y\|, \quad \forall t \in (0,1)$$

or, equivalently:

$$\frac{f(z) - f(x)}{\|z - x\|} + \frac{f(z) - f(y)}{\|z - y\|} \leq \varepsilon, \quad \forall z \in (x, y).$$

Caution: The term "approximately convex" is also used for a long time for another type of functions, namely the ε-convex functions in the sense of Hyers-Ulam (1952).

Examples of approximately convex functions

- Convex functions

- Semiconvex functions

- Strongly paraconvex functions

- Strictly differentiable functions

- (Daniilidis-Georgiev, 2004) *In* $\mathbb{R}^n$*, continuous approximately convex functions coincide with lower-*C^1 *functions*, that is, functions locally representable as a maximum of a compactly indexed family of C^1 functions

Properties of convex functions shared by approximately convex functions

• (Ngai-Luc-Théra, 2000) A lower semicontinuous approximately convex function is locally Lipschitz on the interior of its domain

• (Ngai-Luc-Théra, 2000) If $f : X \to \mathbb{R} \cup \{+\infty\}$ is approximately convex at $x_0 \in X$, then $\partial_C f(x_0) = \partial_F f(x_0)$

• (Daniilidis-Jules-Lassonde, 2009; Georgiev, 1997, for the continuous case) If $f : X \to \mathbb{R} \cup \{+\infty\}$ is directionally approximately convex at $x_0 \in X$, then $\partial_C f(x_0) = \partial_H f(x_0)$

• (Zajíček, 2008) For approximately convex functions, Fréchet differentiability coincide with strict differentiability

• (Zajíček, 2008) In Asplund spaces, continuous approximately convex functions are generically Fréchet differentiable on the interior of their domain

Memo: Basic subdifferentials

Clarke:

$$\partial_C f(x) := \{x^* \in X^* \mid \langle x^*, h\rangle \leq f'_{CR}(x;h),\ \forall h \in X\},$$

where

$$f'_{CR}(x;h) := \sup_{\lambda>0} \limsup_{\substack{t\searrow 0\\ x'\to_f x}} \inf_{h'\in B(h,\lambda)} \frac{f(x'+th')-f(x')}{t}.$$

Hadamard:

$$\partial_H f(x) := \{x^* \in X^* \mid \langle x^*, h\rangle \leq \liminf_{\substack{t\searrow 0\\ h'\to h}} \frac{f(x+th')-f(x)}{t},\ \forall h \in X\}.$$

Fréchet:

$$\partial_F f(x) := \{x^* \in X^* \mid \liminf_{\|h\|\to 0} \frac{f(x+h)-f(x)-\langle x^*, h\rangle}{\|h\|} \geq 0\}$$

Always:

$$\partial_F f(x) \subset \partial_H f(x) \subset \partial_C f(x).$$

Submonotone operators

A set-valued operator $T : X \rightrightarrows X^*$ is said to be *submonotone* (Spingarn, 1981, Janin, 1982, Daniilidis-Georgiev, 2004) (resp., *directionally submonotone* (Georgiev, 1997)) *at* $x_0 \in X$, if:

For every $\varepsilon > 0$ (resp., and $d \in S_X$) there exists $\delta > 0$ such that for all $x, y \in B(x_0, \delta)$ (resp., with $x \neq y$ and $(x - y)/\|x - y\| \in B(d, \delta)$) and all $x^* \in T(x)$ and $y^* \in T(y)$, one has

$$\langle x^* - y^*, x - y\rangle \geq -\varepsilon\|x - y\|.$$

Properties of monotone operators shared by submonotone operators

• (Georgiev, 1997) A directionally submonotone operator is locally bounded on the interior of its domain

• (Georgiev, 1997) In Asplund spaces, directionally submonotone operators are generically single-valued and upper semicontinuous on the interior of their domain

Background: subdifferential characterization of convexity

A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is *convex* if for all $x, y \in \operatorname{dom} f$:

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y), \quad \forall t \in (0,1)$$

or, equivalently:

$$\frac{f(z) - f(x)}{\|z - x\|} + \frac{f(z) - f(y)}{\|z - y\|} \leq 0, \quad \forall z \in (x, y).$$

A set-valued operator $T : X \rightrightarrows X^*$ is *monotone* if for all $x, y \in X$ and all $x^* \in T(x)$ and $y^* \in T(y)$:

$$\langle x^* - y^*, x - y \rangle \geq 0.$$

Thm (Correa-Jofré-Thibault) f *convex* $\Longleftrightarrow \partial_C f$ *monotone.*

Proof of $\partial_C f$ monotone $\Longrightarrow$ f convex

- Case f *differentiable* based on the Mean Value Theorem:

MVT: $\forall x, z \in X, x \neq z, \exists \bar{x} \in [x, z) : f(z) - f(x) = \langle f'(\bar{x}), z - x\rangle$

Let $x, y \in X$ and $z \in (x, y)$. Apply MVT on $[x, z)$ and on $(z, y]$:

$$\begin{cases} \frac{f(z)-f(x)}{\|z-x\|} = \langle f'(\bar{x}), \frac{z-x}{\|z-x\|}\rangle, \text{ for some } \bar{x} \in [x, z) \\ \frac{f(z)-f(y)}{\|z-y\|} = \langle f'(\bar{y}), \frac{z-y}{\|z-y\|}\rangle, \text{ for some } \bar{y} \in (z, y] \end{cases}$$

Put $\frac{z-x}{\|z-x\|} = -\frac{\bar{x}-\bar{y}}{\|\bar{x}-\bar{y}\|}$ and $\frac{z-y}{\|z-y\|} = \frac{\bar{x}-\bar{y}}{\|\bar{x}-\bar{y}\|}$ and add:

$$\frac{f(z) - f(x)}{\|z - x\|} + \frac{f(z) - f(y)}{\|z - y\|} = \langle -f'(\bar{x}) + f'(\bar{y}), \frac{\bar{x} - \bar{y}}{\|\bar{x} - \bar{y}\|}\rangle \leq 0.$$

■ Case f *locally Lipschitz*, same proof based on Lebourg Mean Value Theorem:

$$\boxed{\forall x, z \in X, x \neq z, \exists \bar{x} \in [x, z)\ \exists \bar{x}^* \in \partial_C f(\bar{x}) : f(z) - f(x) = \langle \bar{x}^*, z - x \rangle}$$

■ Case $f : X \to \mathbb{R} \cup \{+\infty\}$ *lower semicontinuous*, same proof based on Zagrodny two points mean value inequality:

> *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous. Let $x, z \in X$ with $x \in \operatorname{dom} f$ and $x \neq z$, and let $r \in \mathbb{R}$ such that $r \leq f(z)$. Then, there exist $\bar{x} \in [x, z)$, and sequences $\{(x_n, x_n^*)\}_n \subset \partial_C f$ with $x_n \to_f \bar{x}$ such that*
>
> $$\frac{r - f(x)}{\|z - x\|} \leq \liminf_n \langle x_n^*, \frac{z - x_n}{\|z - x_n\|} \rangle.$$

Subdifferential characterization of approximate convexity

(Daniiilidis-Jules-Lassonde, 2009) *Let* $f : X \to \mathbb{R} \cup \{+\infty\}$ *be lower semicontinuous and let* $x_0 \in X$*. The following are equivalent:*

(i) f *is (directionally) approximately convex at* x_0*;*

(ii) $\partial_C f$ *is (directionally) submonotone at* x_0*.*

Remarks. 1. For locally Lipschitz functions, the result was proved by Daniilidis-Georgiev, 2004.

2. The case of approximately convex functions has been independently established by Ngai-Penot, 2007.

3. The result is also valid for any subdifferential in appropriate spaces.

Main tool: A three points mean value inequality

(Daniilidis-Jules-Lassonde, 2009) *Let* $f : X \to \mathbb{R} \cup \{+\infty\}$ *be lower semi-continuous. Let* $x, y \in \operatorname{dom} f$ *with* $x \neq y$*, and let* $z \in (x, y)$ *and* $r \in \mathbb{R}$ *such that* $r \leq f(z)$*. Then, there exist* $\bar{x} \in [x, z)$*,* $\bar{y} \in (z, y]$ *and sequences* $\{(x_n, x_n^*)\}_n \subset \partial_C f$ *with* $x_n \to_f \bar{x}$*, and* $\{(y_n, y_n^*)\}_n \subset \partial_C f$ *with* $y_n \to_f \bar{y}$*, such that*

$$\frac{r - f(x)}{\|z - x\|} + \frac{r - f(y)}{\|z - y\|} \leq \liminf_n \langle x_n^* - y_n^*, \frac{y_n - x_n}{\|y_n - x_n\|} \rangle.$$

Maximal submonotonicity of $\partial_C f$

A set-valued operator $T : X \rightrightarrows X^*$ is said to be *maximal directionally submonotone at* $x_0 \in X$ if it is directionally submonotone at x_0 and there is no operator $S : X \rightrightarrows X^*$ directionally submonotone at x_0 such that $T(x) \subset S(x)$ for every x in some neighborhood of x_0 and $T(x_0) \neq S(x_0)$.

(Daniilidis-Jules-Lassonde, 2009) *Let* $f : X \to \mathbb{R} \cup \{+\infty\}$ *be lower semi-continuous and let* $x_0 \in \operatorname{dom} f$. *If* $\partial_C f$ *is directionally submonotone at* x_0, *then* $\partial_C f$ *is actually maximal directionally submonotone at* x_0.

Remark. The result is also valid for any subdifferential in appropriate spaces.

References

[1] R. Correa, A. Jofré and L. Thibault, Characterization of lower semicontinuous convex functions, *Proc. Amer. Math. Soc.*, **116**, (1992), 67–72.

[2] A. Daniilidis and P. Georgiev, Approximate convexity and submonotonicity, *J. Math. Anal. Appl.*, **291**, (2004), 292–301.

[3] A. Daniilidis, P. Georgiev, and J.-P. Penot, Integration of multivalued operators and cyclic submonotonicity, *Trans. Amer. Math. Soc.*, **355**, (2003), 177–195.

[4] A. Daniilidis, F. Jules, and M. Lassonde, Subdifferential characterization of approximate convexity: the lower semicontinuous case, *Math. Programming B*, **116**/1-2, (2009), 115-127

[5] P. Georgiev, Submonotone Mappings in Banach Spaces and Applications, *Set-Valued Analysis*, **5**, (1997), 1-35.

[6] R. Janin, Sur des multiapplications qui sont des gradients généralisés, *C.R. Acad. Sc. Paris*, **294**, (1982), 117–119.

[7] H.V. Ngai, D.T. Luc, and M. Théra, Approximate convex functions, *Journal of Nonlinear and Convex Analysis*, **1**, (2000), 155–176.

[8] H.V. Ngai and J.-P. Penot, Approximately convex functions and approximately monotone operators, *Nonlinear Anal.*, **66**, (2007), 547–564.

[9] J.E. Spingarn, Submonotone subdifferentials of Lipschitz functions, *Trans. Amer. Math. Soc.*, **264**, (1981), 77-89.

[10] L. Zajíček, Differentiability of approximately convex, semiconcave and strongly paraconvex functions, *J. Convex Anal.*, **15** (2008), no. 1, 1–15.