2nd Summer School on Generalized Convex Analysis

NSYSU, Kaohsiung, Taiwan ROC

Lecture # 3:

Fixed points and minimax

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July 15 to July 19, 2008

BASIC INTERSECTION THEOREMS

Sperner (1928): *In* $\mathbb{R}^n$ *with unit vectors* $\{ e_i \mid i \in I \}$, *let* n *closed sets* $\{ A_i \mid i \in I \}$ *satisfy:*

The union of the n *sets contains the simplex* conv$\{ e_i \mid i \in I \}$ *and each set* A_i *contains the* $\hat{\imath}$*-face* conv$\{ e_j \mid j \in I, j \neq i \}$.

Then all the sets have a point in common.

Knaster-Kuratowski-Mazurkiewicz (1929): *In* $\mathbb{R}^n$ *with unit vectors* $\{ e_i \mid i \in I \}$, *let* n *closed sets* $\{ A_i \mid i \in I \}$ *satisfy:*

For every $J \subset I$ *the* J*-face* conv$\{ e_j \mid j \in J \}$ *is contained in the union* $\bigcup \{ A_j \mid j \in J \}$.

Then all the sets have a point in common.

Klee (1951): *In a topological vector space, let* n *closed convex sets satisfy:*

The union of the n *sets is convex and the intersection of every* $n - 1$ *of them is nonempty.*

Then all the sets have a point in common.

BASIC INTERSECTION THEOREMS REVISITED

(replacing $\mathbb{R}^n$ with unit vectors $\{ e_i \mid i \in I \}$ by a TVS with an arbitrary family of (not necessarily distinct) points $\{ x_i \mid i \in I \}$)

Basic conditions implying that n sets have a point in common:

Sperner/Alexandroff-Pasynkoff (1957): *In a TVS, n closed sets $\{ A_i \mid i \in I \}$ with n points $\{ x_i \mid i \in I \}$ such that:*
The union of the n sets contains the convex hull $\operatorname{conv}\{ x_i \mid i \in I \}$ *and each set A_i contains the convex hull* $\operatorname{conv}\{ x_j \mid j \in I, j \neq i \}$.

KKM/Ky Fan (1961): *In a TVS, n closed sets $\{ A_i \mid i \in I \}$ with n points $\{ x_i \mid i \in I \}$ such that:*
For every $J \subset I$ the convex hull $\operatorname{conv}\{ x_j \mid j \in J \}$ *is contained in* $\bigcup\{ A_j \mid j \in J \}$.

Klee (1951): *In a TVS, n closed convex sets with:*
The union of the n sets is convex and the intersection of every $n-1$ of them is nonempty.

COMMENTS

- Passage from the simplex version to the TVS version is direct

- KKM $\Rightarrow$ Sperner $\Rightarrow$ Klee

- For convex sets, all the results are equivalent:

  KKM for convex sets $\Leftrightarrow$ Sperner for convex sets $\Leftrightarrow$ Klee

- For convex sets, proofs are easy

- For arbitrary (closed) sets, no elementary proof of KKM Theorem is known; original proof relies on Sperner combinatorial lemma

- All the results hold for open sets as well

APPLICATIONS OF THE CONVEX VERSIONS: MINIMAX (IN)EQUALITIES

Ky Fan minimax inequality (1972) (special case): *Let K be a nonempty convex compact set in a TVS and let $\varphi : K \times K \to \mathbb{R}$ such that:*

(1) *For each $x \in K$, the function $y \mapsto \varphi(x, y)$ is quasi-concave,*
(2) *For each $y \in K$, the function $x \mapsto \varphi(x, y)$ is quasi-convex lsc,*
(3) *For each $x \in K$, $\varphi(x, x) \leq 0$.*

Then, there exists $\bar{x} \in K$ such that $\varphi(\bar{x}, y) \leq 0$ for all $y \in K$, i.e.,

$$\min_{x \in K} \sup_{y \in K} \varphi(x, y) \leq 0.$$

Sion minimax equality (1958): *Let C and D be two nonempty convex sets in TVS', one of them being compact, and let $f : C \times D \to \mathbb{R}$ such that $f(., y)$ is quasi-convex lsc and $f(x, .)$ is quasi-concave usc. Then:*

$$\inf_{x \in C} \sup_{y \in D} f(x, y) = \sup_{y \in D} \inf_{x \in C} f(x, y).$$

APPLICATIONS OF THE CONVEX VERSIONS: VARIATIONAL INEQUALITIES

Definitions. Let X be a TVS, X^* its topological dual and $T : X \rightrightarrows X^*$ a set-valued operator. We say that T is *monotone* if

$$\forall (x, x^*) \in gr(T),\ \forall (y, y^*) \in gr(T) : \langle y^* - x^*, y - x \rangle \geq 0$$

and *quasi-monotone* if

$$\forall (x, x^*) \in gr(T),\ \forall (y, y^*) \in gr(T) : \langle x^*, y - x \rangle > 0 \Rightarrow \langle y^*, y - x \rangle \geq 0.$$

Debrunner-Flor (1964) (special case)-Minty (1967): *Let $K \subset X$ be a nonempty convex compact set in a TVS and let $T : X \rightrightarrows X^*$ be monotone with* dom $T \subset K$*. Then, there exists $\bar{y} \in K$ such that*

$$\langle x^*, x - \bar{y} \rangle \geq 0, \quad \forall (x, x^*) \in gr(T).$$

Aussel-Hadjisavvas (2004): *Let $K \subset X$ be a nonempty convex compact set in a TVS and let $T : X \rightrightarrows X^*$ be quasi-monotone with* dom $T = K$*. Moreover, assume that T is hemicontinuous from X into (X^*, w^*) with convex w^*-compact values. Then, there exists $\bar{y} \in K$ et $\bar{y}^* \in T\bar{y}$ such that*

$$\langle \bar{y}^*, x - \bar{y} \rangle \geq 0, \quad \forall x \in K .$$

EQUIVALENT FORMULATIONS OF THE KKM THEOREM: FIXED POINT AND MINIMAX

Fan (1961)-Browder (1968): *Let K be a nonempty convex compact set in a TVS and let $\psi : K \rightrightarrows K$ a set-valued map verifying:*
(1) For each $x \in K$, the set $\psi(x)$ is convex,
(2) For each $y \in K$, the set $\psi^{-1}(y)$ is open in K.
Then, there exists $\bar{x} \in K$ such that $\psi(\bar{x}) = \emptyset$ or $\bar{x} \in \psi(\bar{x})$.

Ky Fan minimax inequality (1972): *Let K be a nonempty convex compact set in a TVS and let $\varphi : K \times K \to \mathbb{R}$ such that:*
(1) For each $x \in K$, the function $y \mapsto \varphi(x, y)$ is quasi-concave,
(2) For each $y \in K$, the function $x \mapsto \varphi(x, y)$ is lsc,
(3) For each $x \in K$, $\varphi(x, x) \leq 0$.
Then, there exists $\bar{x} \in K$ such that $\varphi(\bar{x}, y) \leq 0$ for all $y \in K$, i.e.,

$$\min_{x \in K} \sup_{y \in K} \varphi(x, y) \leq 0.$$

Brouwer (1909): *Every continuous self-map $f : K \to K$ of a nonempty convex compact subset of $\mathbb{R}^n$ possesses a fixed point.*

EQUIVALENT FORMULATIONS OF THE KKM THEOREM: VARIATIONAL INEQUALITIES

Debrunner-Flor (1964)-Browder (1967): *Let* $K \subset X$ *be a nonempty convex compact set in a TVS,* $T : X \rightrightarrows X^*$ *be monotone with* $\operatorname{dom} T \subset K$ *and* $u : K \to X^*$ *be continuous. Then, there exists* $\bar{y} \in K$ *such that*

$$\langle u(\bar{y}) + x^*, x - \bar{y} \rangle \geq 0, \quad \forall (x, x^*) \in gr(T).$$

Hartman-Stampacchia (1966)-Browder (1968): *Let* $K \subset X$ *be a nonempty convex compact set in a TVS and let* $T : X \rightrightarrows X^*$ *be monotone with* $\operatorname{dom} T = K$ *and* $u : K \to X^*$ *be continuous. Moreover, assume that* T *is hemicontinuous from* X *into* (X^*, w^*) *with convex* w^**-compact values. Then, there exists* $\bar{y} \in K$ *et* $\bar{y}^* \in T\bar{y}$ *such that*

$$\langle u(\bar{y}) + \bar{y}^*, x - \bar{y} \rangle \geq 0, \quad \forall x \in K .$$

EQUIVALENT FORMULATIONS OF THE KKM THEOREM: TOPOLOGICAL KLEE-TYPE INTERSECTION THEOREM

Definition. A topological space X is said to be *contractible* provided there exists a continuous map $f : X \times [0,1] \to X$ such that $f(1,.)$ is the identity map and $f(0,.)$ is the constant map.

> Horvath-Lassonde (1997): *In a TVS, let n closed convex sets such that:*
>
> *The union of the n sets is contractible and the intersection of every $n-1$ of them is nonempty.*
>
> *Then all the sets have a point in common.*

Proof of the equivalence.

Topolological Klee-type Theorem

$\Rightarrow$ *The n-sphere is not contractible*

$\Rightarrow$ Brouwer's Fixed Point Theorem

$\Rightarrow$ KKM Theorem

$\Rightarrow$ Topolological Klee-type Theorem !!

COMPETITIVE FAN-BROWDER FIXED POINT THEOREM

Notation. Let I be an arbitrary set of indices. Given a family $\{X_i \mid i \in I\}$ of topological spaces, we let $X = \prod_{i \in I} X_i$ be the topological product of the spaces and $\pi_i : X \to X_i$ be the projection of X onto X_i. We simply write $\pi_i(x) = x_i$.

Toussaint (1984): *For each $i \in I$, assume:*

(1) *X_i is a nonempty convex compact set in a TVS;*

(2) *$\varphi_i : X \rightrightarrows X_i$ has convex values and open fibers.*

Then, the following alternative holds:

(A) *There exist $i \in I$ and $\bar{x} \in X$ such that $\bar{x}_i \in \varphi_i(\bar{x})$, or,*

(B) *There exists $\bar{x} \in X$ such that, for each $i \in I$, $\varphi_i(\bar{x}) = \emptyset$.*

Remark. For finite dimensional spaces X_i and finite I, the first theorem of this type is due to Gale and Mas-Colell (1975) (see also Debreu (1950)).

PROOF OF THE COMPETITIVE FAN-BROWDER FIXED POINT THEOREM

Suppose (B) is not verified, that is, $X = \bigcup\{\text{dom}\,\varphi_i \mid i \in I\}$ where $\text{dom}\,\varphi_i = \{x \in X \mid \varphi_i(x) \neq \emptyset\}$ is open. Since X is compact, there exist a finite subset $J \subset I$ and, for each $j \in J$, a closed subset $A^j \subset \text{dom}\,\varphi_j$ so that $X = \bigcup\{A^j \mid j \in J\}$. Consider the map $\varphi : X \rightrightarrows X$ defined by

$$\varphi(x) = \{y \in X \mid y_j \in \varphi_j(x)\ \forall\, j \in J(x)\},$$

where $J(x) = \{j \in J \mid x \in A^j\}$ is finite and not empty. It is easily verified that φ has nonempty convex values and open fibers, so by Fan-Browder's Theorem there is $\bar{x} \in X$ such that $\bar{x} \in \varphi(\bar{x})$, in particular there is $j \in J$ such that $\bar{x}_j \in \varphi_j(\bar{x})$, that is, (A) is verified.

■

NASH EQUILIBRIA

Notation. For $x \in X$, $i \in I$ and $y_i \in X_i$, we denote by (x^i, y_i) the point in X with the same coordinates as x except the i-th which is replaced by y_i.

Nash (1950): *For each $i \in I$, assume:*

(1) *X_i is a nonempty convex compact set in a TVS;*

(2) *$u_i : X \to \mathbb{R}$ is continuous and, for each $x \in X$, $y_i \mapsto u_i(x^i, y_i)$ is quasi-concave.*

Then, there exists $\bar{x} \in X$ verifying:

$$\forall i \in I, \forall y_i \in X_i, \quad u_i(\bar{x}^i, y_i) \leq u_i(\bar{x}).$$

Proof. Apply the competitive Fan-Browder Theorem with $\varphi_i : X \rightrightarrows X_i$ defined by $\varphi_i(x) = \{y_i \in X_i \mid u_i(x^i, y_i) > u_i(x)\}$ for each $x \in X$. ∎

APPROXIMATION OF USC MULTI-MAP AND KAKUTANI-FAN-GLICKSBERG THEOREM

Approximation: *Let X be a paracompact space, Y a LCTVS and $\varphi : X \rightrightarrows Y$ a USC map with compact convex values. Then, for any neighborhood $\mathcal{U}$ of $gr(\varphi)$ in $X \times Y$, there exists a map $\varphi' : X \rightrightarrows Y$ with open graph and convex values such that $gr(\varphi) \subset gr(\varphi') \subset \mathcal{U}$.*

Kakutani (1941), Ky Fan (1952), Glicksberg (1952): *Let K be a nonempty convex compact set in a LCTVS and $\varphi : K \rightrightarrows K$ a USC map with nonempty compact convex values. Then, there exists $\bar{x} \in K$ such that $\varphi(\bar{x}) = \emptyset$ or $\bar{x} \in \varphi(\bar{x})$.*

Proof. Suppose φ has no fixed point, that is, $gr(\varphi)$ is contained in the open set $\Delta^c = \{(x, y) \mid x \neq y\}$. So there is $\varphi' : K \rightrightarrows K$ with open fibers and convex values such that $gr(\varphi) \subset gr(\varphi') \subset \Delta^c$. Since φ' has no fixed point, by Fan-Browder Theorem there is $\bar{x} \in K$ such that $\varphi'(\bar{x}) = \emptyset$, hence also $\varphi(\bar{x}) = \emptyset$. ∎

CONSTRAINED KAKUTANI-FAN-GLICKSBERG FIXED POINT THEOREM

Let K be a nonempty convex compact set in a LCTVS X. Assume:

(1) $\varphi : K \rightrightarrows K$ *is USC with nonempty compact convex values;*

(2) $\psi : K \rightrightarrows K$ *has open fibers, convex values and no fixed point;*

(3) $V = \{x \in K \mid \varphi(x) \cap \psi(x) \neq \emptyset\}$ *is open.*

Then, there exists $\bar{x} \in \varphi(\bar{x})$ *such that* $\varphi(\bar{x}) \cap \psi(\bar{x}) = \emptyset$.

CONSTRAINED MINIMAX INEQUALITY

Let K be a nonempty convex compact set in a LCTVS X. Assume:

(1) $\varphi : K \rightrightarrows K$ *is USC with nonempty compact convex values;*

(2) $f : K \times K \to \mathbb{R}$ *verifies*

$$\begin{cases} \forall\, y \in K, & x \mapsto f(x,y) \text{ is lsc,} \\ \forall\, x \in K, & y \mapsto f(x,y) \text{ is quasi-concave,} \\ \forall\, x \in K, & f(x,x) \le 0; \end{cases}$$

(3) *The set* $\left\{ x \in K \mid \sup_{y \in \varphi(x)} f(x,y) > 0 \right\}$ *is open.*

Then, there exists $\bar{x} \in K$ such that

$$\begin{cases} \bar{x} \in \varphi(\bar{x}), \\ \sup_{y \in \varphi(\bar{x})} f(\bar{x},y) \le 0 \,. \end{cases}$$

Note. In case $f(x,y) = \langle Ax, x - y \rangle$ with $A : K \to X^*$, the constrained minimax inequality is called *quasi-variational inequality*.