An Overview of Generalized Convexity

Dinh The Luc

University of Avignon, Avignon

Outline

- ▷ Brief history of convex analysis
- ▷ From convexity to generalized convexity
- ▷ Quasiconvex functions
- ▷ Pseudoconvex functions
- ▷ Invex functions
- ▷ Extensions of GC functions
- ▷ Generalized monotone operators
- ▷ References

Brief history of convex analysis

- ♣ Ancient times : triangles, circles, polygons
- ♣ Beginning of the 20th century (Brunn, Minkowski, Hermite, Hadamard, Farkas, Steinitz, Jensen ...)
  - ▷ Convex bodies
  - ▷ Convex sets
  - ▷ Linear inequalities
  - ▷ Convex functions
- ♣ From the middle of the 20th century (Choquet, Klein, Milman, Hahn-Banach, Fenchel, Moreau, Rockafellar...) :
  - ▷ Convex sets, separation
  - ▷ Convex functions (continuity, differentiability, conjugate, subdifferential)
  - ▷ Convex inequalities, duality etc.

Brief history of convex analysis

Attention : the same terminology in complex analysis

A function f of a complex variable is called convex if

$$\mathrm{Re}\frac{(zf'(z))'}{f'(z)} > 0$$

(the image of a convex domain is convex).

From convexity to generalized convexity

Convex sets :

$$A \subseteq \mathbb{R}^n \text{ is said to be convex} \Leftrightarrow [a,b] \subseteq A \ \forall a,b \in A.$$

What happens if $[a,b]$ is substituted by an arc from a family of functions ?

- Two segments with a common fixed end-point $\Longrightarrow$ star-shaped sets
- Geodesic arcs $\Longrightarrow$ convex geodesic sets.
- Arcs parallel to coordinate axes $\Longrightarrow$ orthogonal convex sets.

From convexity to generalized convexity

Convex functions :

$$f : A \subseteq \longrightarrow \mathbb{R} \text{ is said to be convex} \Leftrightarrow$$

either of the following conditions holds

- Epi $f = \{(x,t) : x \in A, t \geq f(x)\}$ is convex
- $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$
- $f(y) - f(x) \geq f'(x)(y - x)$ when f is differentiable.

From convexity to generalized convexity

What happens if

- Epi f is not convex, but a generalized convex set ? (not so much studied except perhaps for IAR functions).
- Epi f is not convex but Lev(f) are convex ? $\Longrightarrow$ quasi-convex functions.
- $f'(x)(y-x)$ is substituted by a function of (x,y) ? $\Longrightarrow$ invex functions.
- $f(tx+(1-ty))$ is substituted by $f(y)-t\alpha$ for some $\alpha > 0$ and for all t sufficiently small $(f(y) \geq f(x))$ $\Longrightarrow$ pseudo-convex functions.
- $f(tx+(1-t)y)$ is substituted by the minimum of f over the integers whose coordinates are within unit distance from $tx+(1-t)y \Longrightarrow$ discretely convex functions.

From convexity to generalized convexity

Why do we need generalized convex functions ?

- Economics (analyse of microstructure, dislocation structure, demand functions, Cob-Douglas functions, equilibrium, noncooperative game ....)
- Mechanics (frictionless contact)
- Optimization (fractional programming, stochastic programming, multiobjective programming ...)
- Other areas of applied mathematics (statistics, differential equations, Hamilton-Jacobi equations, viscosity ...)

From convexity to generalized convexity

Who have initialized the study of generalized convex functions ? And recently ?

- Von Neumann in 1929 with minimax theorem (according to Guerraggio and Molho)
- De Finetti in 1949.
- Arrow, Avriel, Crouzeix, Demianov, Enthoven, Ferland, Karamardian, Mangasarian, Martos, Rubinov, Schaible, Ziemba, Zang ....in the sixties and seventies of the last century.
- WGGC members and others.

Quasiconvex functions

A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be quasiconvex if either of the following equivalent conditions holds

(1) Sublevel sets of f are convex

(2) $f(tx + (1-t)y) \leq \max\{f(x), f(y))\}$

Quasiconvex functions

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Other variants

- strictly quasiconvex functions (semi-strictly quasiconvex functions)

$$\text{(2) is strict when } x \neq y$$

- semi-strictly quasiconvex functions (explicitly quasiconvex functions)

$$\text{(2) is strict when } \; f(x) \neq f(y)$$

Quasiconvex functions

Characterizations via derivatives (Arrow-Enthoven, 1961)

$$f \text{ is QC} \iff \nabla f \text{ is QM}$$

$$\text{QM} : \max\{\langle \nabla f(x), x - y\rangle, \langle \nabla f(y), y - x\rangle\} \geq 0$$

Equivalent conditions

(1) $f(y) < f(x)$ implies $\langle \nabla f(x), x - y\rangle\} \geq 0$

(2) $f(y) \leq f(x)$ implies $\langle \nabla f(x), x - y\rangle\} \geq 0$

(3) $f(y) < f(x)$ and $\langle \nabla f(x), x - y\rangle = 0$ imply $f(x) \leq f(x + t(x - y)), \forall t > 0.$

Quasiconvex functions

Characterizations via directional derivatives (Komlosi, Luc ....)

$$f \text{ is QC} \Longleftrightarrow f'(.;.) \text{ is QM}$$

Here : $f'(x;u)$ is the directional derivative of f at x in direction u.
f' may be

- Dini lower, upper directional derivatives for f lower semi-continuous on segments
- Clarke-Rockafellar's directional derivatives for f lower semicontinuous.

QM : $\max\{f'(x, x-y), f'(y, y-x)\} \geq 0$

Quasiconvex functions

Characterizations via generalized derivatives (Elleia-Hassouni, Luc, Penot-Quang, Aussel-Lassonde...)

$$f \text{ is QC} \iff \partial f(.;.) \text{ is QM}$$

Here : ∂f is a subdifferential of f.
∂ may be

- Clarke's subdifferential for lower semi-continuous functions ;
- Pseudo-Jacobian for continuous functions.

QM : $\max\{\langle x^*, x-y\rangle, \langle y^*, y-x\rangle\} \geq 0$
for $x^* \in \partial f(x)$ and $y^* \in \partial f(y)$
PROOF. Mean-value theorems.

Quasiconvex functions

Characterizations via normal cones(Aussel, Crouzeix, Danidiilis, Hadjisavvas, Lassonde ...)
(f is continuous, or the space admits a Gâteaux smooth renorm)

$$f \text{ is QC} \Longleftrightarrow N_f \text{ is QM.}$$

Here N_f is Clarke's normal cone to sublevel sets of f
QM : $\max\{\langle x^*, x - y\rangle, \langle y^*, y - x\rangle\} \geq 0$
for $x^* \in \partial f(x)$ and $y^* \in \partial f(y)$

Quasiconvex functions

Characterizations via normal cones(Aussel, Crouzeix, Daniliidis, Hadjisavvas, Lassonde ...)
Clarke's tangent cone

$$T^{\circ}(C;x) = \liminf_{y \in C \to x; t \downarrow 0} \frac{C-y}{t}$$

i.e. $v \in T \circ (C;x)$ iff $\forall t_n \downarrow 0, y_n \in C \to x$ there are $x_n \in C$ such that $v = \lim(x_n - y_n)/t_n$.
Clarke's normal cone

$$N^{\circ}(C;x) = \{x^* : \langle x^*, v \rangle \leq 0, \forall v \in T \circ (C;x)\}.$$

Quasiconvex functions

Quasiconvex subdifferential(Greenberg-Pierskalla, Crouzeix, Daniilidis, Hadjisavvas, Martinez-Legaz, Sach...)

$$\partial^{qcv} f(x) = \partial f(x) \cap N(f;x)$$

Here

$N(f;x)$ is the convex normal cone to the sublevel set of f at x, i.e. $x^* \in N(f;x)$ iff $\langle x^*, y-x\rangle \leq 0, \forall y$ with $f(y) \leq f(x)$.

$\partial f(x)$ is any subdifferential satisfying the mean-value inequality :

$f(y) > f(x) \Longrightarrow \exists x_n \to z \in [x,y), x_n^* \in \partial f(x_n)$ such that $\langle x_n^*, z + t(y-z) - x_n\rangle > 0 \forall t$.

Quasiconvex functions

Quasiconvex subdifferential(Greenberg-Pierskalla, Crouzeix, Daniilidis, Hadjisavvas, Martinez-Legaz, Sach...)

$$\partial^{qcv} f(x) = \partial f(x) \cap N(f; x)$$

Utility of quasiconvex subdifferential
THEOREM : If f is lsc, radially continuous, then the following conditions are equivalent
(i) f is quasi-convex
(ii) $\partial^{qcv} f = \partial f$
(iii) $\partial^{qcv} f$ satisfies the mean-value inequality
(iv) The domain of ∂^{qvc} is dense in the domain of f.

Quasiconvex functions

Quasiconvex conjugate
(Martinez-Legaz, Penot, Rubinov-Dutta, Singer, Volle ...)

c-conjugate for $f : \mathbb{R}^n \to \overline{\mathbb{R}}$:

$$f^c(x^*, t) = -\inf\{f(x) : \langle x, x^* \rangle \geq t\}$$

c'-conjugate for $g : \mathbb{R}^n \times \mathbb{R} \to \overline{\mathbb{R}}$:

$$g^{c'}(x) = -\inf\{g(x^*, t) : \langle x, x^* \rangle \geq t\}$$

Quasiconvex functions

Quasiconvex conjugate
(Martinez-Legaz, Penot, Rubinov-Dutta, Singer, Volle ...)

Here "c" is referred to a special coupling function from an abstract scheme of level set conjugation :

$$c(x, x^*, t) = \begin{cases} 0 & \text{if } \langle x, x^* \rangle \geq t \\ -\infty & \text{otherwise} \end{cases}$$

Quasiconvex functions

Quasiconvex conjugate
(Martinez-Legaz, Penot, Rubinov-Dutta, Singer, Volle ...)

THEOREM : f is evenly quasiconvex $\iff f = f^{cc'}$.
In particular, every evenly quasiconvex function is the pointwise supremum of a collection of evenly quasiaffine functions (quasiconvex and quasiconcave)

Quasiconvex functions

Quasiconvex conjugate
(Martinez-Legaz, Penot, Rubinov-Dutta, Singer, Volle ...)

f is evenly quasiconvex if its sublevel sets are intersections of open halfspaces.
Lower semicontinuous quasiconvex functions and upper semicontinuous quasiconvex functions are evenly quasiconvex.

Pseudoconvex functions

A differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be pseudoconvex if either of the following equivalent conditions holds

(1) $f(y) < f(x) \Longrightarrow \langle \nabla f(x), y - x \rangle < 0$

(2) $\langle \nabla f(x), y - x \rangle > 0 \Longrightarrow \langle \nabla f(y), y - x \rangle > 0$

(3) $\langle \nabla f(x), y - x \rangle \geq 0 \Longrightarrow \langle \nabla f(y), y - x \rangle \geq 0$

Pseudoconvex functions

Other variants

- strictly pseudoconvex functions
  $x \neq y$ and $f(y) \leq f(x) \Longrightarrow \langle \nabla f(x), y - x \rangle < 0$
- nondifferentiable pseudoconvex functions
  $f(y) > f(x) \Longrightarrow \exists \beta(x,y) > 0, \delta(x,y) \in (0,1]$ such that

$$f(y) - f(tx + (1-t)y) \geq t\beta(x,y) \forall t \in (0, \delta(x,y))$$

- nondifferentiable strictly pseudoconvex functions
  Inequality holds whenever $f(y) \geq f(x), x \neq y$.

Pseudoconvex functions

THEOREM : f is a differentiable function.

- If f is pseudoconvex, then f is quasiconvex and attains a global minimum at every x with $\nabla f(x) = 0$.
- If f is quasiconvex and attains a local minimum at any x with $\nabla f(x) = 0$, then f is pseudoconvex.
- f is pseudoconvex $\Longleftrightarrow$ ∇f is PM (pseudomonotone).

Here ∇f is PM if

$$\max\{\langle \nabla f(x), x - y\rangle, \langle \nabla f(x), x - y\rangle\} > 0$$

when both of two terms are nonzero.

Pseudoconvex functions

Nondifferentiable pseudoconvex functions

THEOREM :

f is pseudoconvex $\Longleftrightarrow$ $f^+(.;.)$ is PM.

Here

- $f^+(x;u)$ is Dini upper directional derivative of f at x in direction u ;
- f^+ is PM if

$$\max\{f^+(x; x-y), f^+(y; y-x)\} > 0$$

provided both of the two terms are nonzero.

Pseudoconvex functions

Nondifferentiable pseudoconvex functions

THEOREM :

♣ f is pseudoconvex $\Longrightarrow$

(1) ∂f is QM

(2) $f(x) < f(y) \Rightarrow \exists y^* \in \partial f(y) : \langle y^*, x - y \rangle < 0.$

♣ f is pseudoconvex $\Longleftarrow \partial f$ is PM.

Here

- ∂f may be Clarke's subdifferential when f is locally Lipschitz and Pseudo-Jacobian when f is continuous.
- ∂f is PM if $\max\{\langle x*, x - y \rangle, \langle y^*, y - x \rangle\} > 0$ with $x^* \in \partial f(x), y^* \in \partial f(y)$ provided both of the two terms are nonzero.

Pseudoconvex functions

Nondifferentiable pseudoconvex functions (lower semicontinuous and continuous on segments)
REMARK :
If one modifies the definition of pseudoconvexity by using a given subdifferential :
f is pseudoconvex if $\exists x^* \in \partial f(x) : \langle x*, y - x \rangle \geq 0 \Longrightarrow f(z) \leq f(y) \forall z \in [x, y]$, then

$$f \text{ is pseudoconvex} \iff \partial f \text{ is PM}$$

where ∂f is a subdifferential satisfying a mean value inequality.

Invex functions

f is differentiable (Hanson, Craven)
DEFINITION : f is invex if there is a vector function η of two variables such that

$$f(x) - f(y) \geq \langle \nabla f(y), \eta(x, y) \rangle$$

THEOREM :
f is invex $\Longleftrightarrow$ every x with $\nabla f(x) = 0$ is a global minimum point.

Extensions of GC functions

Vector quasiconvex functions :
DEFINITION : $f : \mathbb{R}^n \to \mathbb{R}^m$ is QC if its sublevel sets are convex.
Other variants

- QC : $f(x) \leq f(y) \Longrightarrow f(tx + (1-t)y) \leq f(y)$ (Cambini, Martein ...)
- Semistrictly quasiconvex functions, strictly quasiconvex functions.
- Set-valued quasiconvex maps.

Extensions of GC functions

Vector quasiconvex functions :
THEOREM : f is QC $\Longleftrightarrow$ $\lambda \circ f$ is QC for all extreme directions λ of the ordering cone.

(Luc : polyhedral cone
Benoist-Borwein-Popovici : convex cone)

Extensions of GC functions

REMARK :

- Similar extensions for pseudoconvex, invex functions.
- Several applications in vector optimization (existence, optimality conditions, local-global property, topological properties of solution sets ...),

Generalized monotone operators

Monotone operators :
DEFINITION : $F : E \to E'$ is monotone if

$$\langle F(x), x - y\rangle + \langle F(y), y - x\rangle \leq 0$$

EXAMPLE : Linear positive semi-definite operators

Generalized monotone operators

Monotone operators :
DEFINITION : $F : E \rightarrow E'$ is monotone if

$$\langle F(x), x - y \rangle + \langle F(y), y - x \rangle \leq 0$$

IMPORTANT FEATURES :

$\diamondsuit$ (Kachurovskii) The derivative of a convex function is monotone.

$\diamondsuit$ Equation $Ax = b$ has a solution in a separable, reflexive B-space for every b if

- A is monotone
- A is continuous on finite dimensional subspaces
- A is coercive : $\langle Ax, x \rangle / \|x\| \rightarrow \infty$ as $\|x\| \rightarrow \infty$.

Generalized monotone operators

Generalization of monotone operators :

$\diamondsuit$ The first feature of monotone operators

$$\Downarrow$$

Generalized monotonicity of the derivative of generalized convex functions

- quasimonotone/ semistrictly quasimonotone/ strictly quasimonotone operators
- pseudomonotone/ strictly pseudomonotone operators

Generalized monotone operators

Quasimonotone operator

$$\max\{\langle F(x), x-y\rangle, \langle F(y), y-x\rangle \geq 0$$

♢ f is QC $\Longleftrightarrow \nabla f$ is QM/ CQM
(cyclically quasimonotone :
$\min\{\langle F(x_i), x_{i+1}-x_i\rangle : i=1,...,n; x_{n+1}=x_1\} \leq 0$)

Generalized monotone operators

Characterization of QM operators

$$\max\{\langle F(x), x-y\rangle, \langle F(y), y-x\rangle \geq 0$$

◊ A differentiable operator F is QM $\Longleftrightarrow$

(i) $\langle F(x), u\rangle = 0 \Rightarrow \langle F'(x)u, u\rangle \geq 0$

(ii) $F(x) = 0, F'(x)u = 0 \Rightarrow \forall s > 0, \exists t \in (0, s]$:
$\langle F(x+tu), u\rangle \geq 0.$

PROOF. Differential equation + implicit function theorem (Crouzeix-Ferland ; Luc-Schaible)

COROLLARY : Second-order characterizations of quasiconvex functions (Hessian matrix of f instead of F'.)

INTERESTING CASE : f is a quadratic function (Ferland, Martos, Schaible)

Generalized monotone operators

Generalization of monotone operators :

◊ The second feature of monotone operators

$$\Downarrow$$

Generalized monotonicity for solving inequalities and inclusions

- proper quasimonotone/ pseudomonotone operators
- Brezis' pseudomonotone/ monotone-like operators
- KKM-operators

Generalized monotone operators

- properly quasimonotone :

$$x \in conv\{x_1, ..., x_k\} \Longrightarrow \exists i : \langle F(x_i), x - x_i \rangle \geq 0$$

Other name : diagonally quasiconvex function

APPLICATIONS : Variational inequalities, equilibrium

(Aussel, Flores-Bazan, Hadjisavvas, Luc, Tan,...)

Generalized monotone operators

- Brezis' pseudomonotone operator :
  $\limsup_{x_n \rightharpoonup x} \langle F(x_n), x_n - x \rangle \leq 0 \Longrightarrow$
  $\liminf_{x_n \rightharpoonup x} \langle F(x_n), x_n - y \rangle \geq \langle F(x), x - y \rangle \forall y.$

Particular cases :
1) Monotone and hemicontinuous operators are B-pseudomonotone
2) Strongly continuous operators are B-pseudomonotone
3) Continuous operators in finite dimensional spaces are B-pseudomonotone.

APPLICATIONS : Variational inequalities (Aubin, Brezis, Browder, Minty, Chowdhury, Tan, Yao ....)

Generalized monotone operators

- KKM-relation : $R \subseteq E \times E$ is KKM if

$$a \in conv\{a_1, ..., a_k\} \Longrightarrow \exists i : (a, a_i) \in R$$

Particular cases :
1) Properly quasimonotone operator
2) Quasiconvex inclusions (Hai-Khanh ; Lin-Chen)
3) KKM mapping
APPLICATIONS : Variational relation problems including quasi-variational inequalities, inclusions, generalized quasi-equilibrium problems ... (Anh, Ansari, Chadli, Li, Lin, Chen, Huang, Hai, Khanh, Tan, Sach...)

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