Abstract Convex Analysis

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2nd Summer School: Generalized Convexity Analysis
Introduction to Theory and Applications

JULY 17, 2008
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$$f : \mathbb{R}^n \to \overline{\mathbb{R}} = [-\infty, +\infty]$$

$$\begin{aligned} f^* &: \mathbb{R}^n \to \overline{\mathbb{R}} \\ f^*(x^*) &= \sup \{\langle x, x^* \rangle - f(x)\} \end{aligned}$$

f^* is convex, l.s.c. and proper.

f^{**} is the largest l.s.c. proper convex minorant of f.

$x^* \in \mathbb{R}^n$ is a subgradient of f at $x_0 \in f^{-1}(\mathbb{R})$ if

$$f(x) - f(x_0) \geq \langle x - x_0, x^* \rangle \qquad (x \in \mathbb{R}^n).$$

$$\partial f(x_0) = \left\{ x^* \in \mathbb{R}^n : \text{is a subgradient of } f : \mathbb{R}^n \to \overline{\mathbb{R}} \text{ at } x_0 \right\}$$

$$x^* \in \partial f(x_0) \text{ if and only if } f(x_0) + f^*(x^*) = \langle x_0, x^* \rangle$$
$$x^* \in \partial f^{**}(x_0) \text{ if and only if } x_0 \in \partial f^*(x^*)$$
$$\partial f(x_0) \neq \emptyset \implies f^{**}(x_0) = f(x_0)$$
$$f^{**}(x_0) = f(x_0) \implies \partial f^{**}(x_0) = \partial f(x_0)$$

$$\left\langle x_1 - x_0, x_0^* \right\rangle + \left\langle x_2 - x_1, x_1^* \right\rangle + \cdots + \langle x_0 - x_m, x_m^* \rangle \leq 0$$

for any set of pairs $\left(x_i, x_i^*\right)$, $i = 0, 1, \ldots, m$ (m arbitrary) such that $x_i^* \in \partial f(x_i)$.

$$\varphi : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$$

$$(\mathcal{P}_u) \qquad \text{minimize } \varphi(x, u)$$

$$(\mathcal{P}) \qquad \text{minimize } \varphi(x, 0)$$

$$\begin{aligned} p &: \mathbb{R}^m \to \overline{\mathbb{R}} \\ p(u) &= \inf_{x \in \mathbb{R}^n} \varphi(x, u) \end{aligned}$$

$$(\mathcal{D}) \qquad \text{maximize } -\varphi^*(0, u^*)$$

$$-\varphi^*(0, u^*) = -p^*(u^*)$$

The optimal value is $p^{**}(0)$.
The set of optimal dual solutions is $\partial p^{**}(0)$.

$$(\mathcal{D}_{x^*}) \qquad \text{maximize } -\varphi^*(x^*, u^*)$$

$$\psi : \mathbb{R}^m \times \mathbb{R}^n \to \overline{\mathbb{R}}$$
$$\psi(u^*, x^*) = \varphi^*(x^*, u^*)$$

$$(\mathcal{D}_{x^*}) \qquad \text{minimize } \psi(u^*, x^*)$$

$$\text{maximize } -\psi^*(0, x)$$

$$f : \mathbb{R}^n \to \overline{\mathbb{R}}, g : \mathbb{R}^n \to \mathbb{R}^m$$

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g(x) \le 0 \end{array}$$

$$\varphi : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$$

$$\varphi(x, u) = \begin{cases} f(x) & \text{if } g(x) + u \le 0 \\ +\infty & \text{otherwise} \end{cases}$$

$$L : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$$
$$L(x, u^*) = f(x) + \langle g(x), u^* \rangle$$

$$\begin{array}{ll} \text{maximize} & \inf_{x \in \mathbb{R}^n} L(x, u^*) \\ \text{subject to} & u^* \ge 0 \end{array}$$

$$f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}, \;\; g : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$$

$$\text{minimize } f(x) - g(x)$$

$$\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \overline{\mathbb{R}}$$

$$\varphi(x, u) = f(x + u) - g(x)$$

$$g_* : \mathbb{R}^n \to \overline{\mathbb{R}}$$

$$g_*(u^*) = -(-g)^*(-u^*) = \inf \{\langle x, u^* \rangle - g(x)\}$$

$$\text{maximize } g_*(u^*) - f^*(u^*)$$

$f, g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ l.s.c. proper convex functions

$$\begin{aligned}
\inf_{x \in \mathbb{R}^n} \{f(x) - g(x)\} &= \inf_{x \in \mathbb{R}^n} \{f(x) - g^{**}(x)\} \\
&= \inf_{x \in \mathbb{R}^n} \{f(x) - \sup_{x^* \in \mathbb{R}^n} \{\langle x, x^* \rangle - g^*(x^*)\}\} \\
&= \inf_{x, x^* \in \mathbb{R}^n} \{f(x) - \langle x, x^* \rangle + g^*(x^*)\} \\
&= \inf_{x^* \in \mathbb{R}^n} \{g^*(x^*) + \inf_{x \in \mathbb{R}^n} \{f(x) - \langle x, x^* \rangle\}\} \\
&= \inf_{x^* \in \mathbb{R}^n} \{g^*(x^*) - f^*(x^*)\}
\end{aligned}$$

GENERALIZED CONVEX CONJUGATION

X, Y arbitrary sets, $c : X \times Y \to \overline{\mathbb{R}}$

$$f : X \to \overline{\mathbb{R}}$$

$$f^c : Y \to \overline{\mathbb{R}}$$
$$f^c(y) = \sup_{x \in X} \{c(x,y) - f(x)\}$$

$$\begin{aligned} +\infty + (-\infty) &= -\infty + (+\infty) = +\infty - (+\infty) \\ &= -\infty - (-\infty) = -\infty \end{aligned}$$

$$c' : Y \times X \to \overline{\mathbb{R}}$$
$$c'(y,x) = c(x,y)$$

$$g : Y \to \overline{\mathbb{R}}$$

$$g^{c'} : X \to \overline{\mathbb{R}}$$
$$g^{c'}(x) = \sup_{y \in Y} \{c(x,y) - g(y)\}$$

c-elementary functions: $\quad x \in X \to c(x,y) - \beta \in \overline{\mathbb{R}}$
c'-elementary functions: $\quad y \in Y \to c(x,y) - \beta \in \overline{\mathbb{R}}$

Φ_c the set of c-elementary functions
$\Phi_{c'}$ the set of c'-elementary functions

Φ set of extended real-valued functions on X

$f : X \longrightarrow \overline{\mathbb{R}}$ is Φ-convex if it is the pointwise supremum of a subset of Φ.

The Φ-convex hull of $f : X \longrightarrow \overline{\mathbb{R}}$: the largest Φ-convex minorant of f.

PROPOSITION. Let $f : X \to \overline{\mathbb{R}}$, $g : Y \to \overline{\mathbb{R}}$, $x \in X$ and $y \in Y$. Then
(i) $f^c(y) \geq c(x,y) - f(x)$, $g^{c'}(x) \geq c(x,y) - g(y)$.
(ii) $f^{cc'c} = f^c$, $g^{c'cc'} = g^{c'}$.
(iii) f^c and $g^{c'}$ are $\Phi_{c'}$-convex and Φ_c-convex, respectively.

PROPOSITION. The Φ_c-convex hull of $f : X \to \overline{\mathbb{R}}$ coincides with $f^{cc'}$.
The $\Phi_{c'}$-convex hull of $g : Y \to \overline{\mathbb{R}}$ coincides with $g^{c'c}$.

COROLLARY. A function $f : X \to \overline{\mathbb{R}}$ is Φ_c-convex if and only if it coincides with $f^{cc'}$.
A function $g : Y \to \overline{\mathbb{R}}$ is $\Phi_{c'}$-convex if and only if it coincides with $g^{c'c}$.

$f : X \to \overline{\mathbb{R}}$ is Φ_c-convex at $x_0 \in X$ if
$f^{cc'}(x_0) = f(x_0)$.
$g : Y \to \overline{\mathbb{R}}$ is $\Phi_{c'}-$convex at $y_0 \in Y$ if
$g^{c'c}(y_0) = g(y_0)$.

EXAMPLE. X, Y dual pair of vector spaces
$\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{R}$ the duality pairing

$$c : X \times Y \to \overline{\mathbb{R}}$$
$$c(x, y) = \log \langle x, y \rangle$$

$f : X \to \overline{\mathbb{R}}$ is Φ_c-convex if and only if e^f is sublinear

EXAMPLE. $X = Y = [0, +\infty]$

$$c : X \times Y \to \overline{\mathbb{R}}$$
$$c(x, y) = xy$$

f: $X \to \overline{\mathbb{R}}$ is Φ_c-convex if and only if
f is convex and nondecreasing.

EXAMPLE. X topological space, Y arbitrary set
$c : X \times Y \to \mathbb{R}$ s.t. $\forall\ (x_0, y, \eta) \in X \times Y \times \mathbb{R}$ and
$\forall\ N$ neighborhood of x_0
$\exists\ y' \in Y$ and $\exists\ N' \subseteq N$, neighborhood of x_0, s.t.

$$c\left(x, y'\right) - c\left(x_0, y'\right) \leq c(x, y) + \eta \qquad (x \in X \backslash N')$$

$$c\left(x, y'\right) - c\left(x_0, y'\right) \leq 0 \qquad (x \in X).$$

$f : X \to \mathbb{R} \cup \{+\infty\}$ is Φ_c-convex if and only if
f is l.s.c. and has a finite-valued c-elementary minorant.
If X is a Hilbert space and $Y = \mathbb{R}_+ \times X$, one can take

$$c : X \times Y \to \mathbb{R}$$
$$c(x, (\rho, y)) = -\rho \|x - y\|^2 .$$

EXAMPLE. $X = Y = \mathbb{R}^n$, $0 < \alpha \leq 1$, $N > 0$

$$\begin{aligned} c &: X \times Y \to \mathbb{R} \\ c(x, y) &= -N \|x - y\|^{\alpha} \end{aligned}$$

$f : X \to \mathbb{R}$ is Φ_c-convex if and only if
f is $\alpha-$Hölder continuous with constant N.

EXAMPLE. X normed space with dual X^*, $0 < \alpha \leq 1$
$Y = \mathcal{B}^*(0, N) \times \mathbb{R}$, $\mathcal{B}^*(0, N)$ the closed ball in X^* with radius $N > 0$

$$\begin{aligned} c &: X \times Y \to \mathbb{R} \\ c(x, (\omega, k)) &= \min\{-(k - \omega(x))^{\alpha}, 0\} + k \end{aligned}$$

($t^{\alpha} = -\infty$ if $t < 0$ and $\alpha \neq 1$)
$f : X \to \mathbb{R}$ is Φ_c-convex if and only if
f is quasiconvex and $\alpha-$Hölder with constant N^{α}.

EXAMPLE. $X = \{0,1\}^n$, $Y = C_1 \times C_2 \times \cdots \times C_n$, with $C_i \subseteq \mathbb{R}$ unbounded from above and from below $c : X \times Y \to \mathbb{R}$ the restriction of the scalar product $f : X \to \overline{\mathbb{R}}$ is Φ_C-convex if and only if either f does not take the value $-\infty$ or $f \equiv -\infty$.

EXAMPLE. $X = \mathbb{R}^n$

$$Y = \bigcup_{k=0}^{n} \left(\mathcal{L}\left(\mathbb{R}^n, \mathbb{R}^k\right) \times \mathbb{R}^k \right) \times \mathbb{R}^n$$

$$\begin{aligned} c \quad &: \quad X \times Y \to \overline{\mathbb{R}} \\ c\left(x, \left(u, z, x^*\right)\right) \quad &= \quad \begin{cases} -\infty & \text{if } u(x) <_L z \\ \langle x, x^* \rangle & \text{if } u(x) = z \\ +\infty & \text{if } u(x) >_L z \end{cases} \end{aligned}$$

$f : X \to \overline{\mathbb{R}}$ is Φ_C-convex if and only if it is convex.

EXAMPLE. $X = Y = \mathbb{R}^n_+$

$$
\begin{aligned}
c \quad &: \quad X \times Y \to \mathbb{R} \\
c\,(x, y) \quad &= \quad \begin{cases} \min_{i \text{ s.t. } y_i > 0} x_i y_i & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}
\end{aligned}
$$

$f : X \longrightarrow \overline{\mathbb{R}}$ is Φ_c-convex if and only if it is ICAR

EXAMPLE. $X = Y = \mathbb{R}^n_{++} \cup \{0\}$

$$
\begin{aligned}
&c : X \times Y \to \mathbb{R} \\
&c\,(x, y) = -\max_{i=1,\ldots,n} x_i y_i
\end{aligned}
$$

$f : X \longrightarrow \overline{\mathbb{R}}$ is Φ_c-convex if and only if it is DCAR.

THEOREM. For any $h : X \longrightarrow \overline{\mathbb{R}}$, the following statements are equivalent:

(i) h is Φ_c-convex.

(ii) $\inf_{x \in X} \{f(x) - h(x)\} = \inf_{x \in X}\{f(x) - h^{cc'}(x)\} \quad \forall\, f$

(iii) $\inf_{x \in X} \{f(x) - h(x)\} = \inf_{y \in Y} \{h^c(y) - f^c(y)\} \quad \forall\, f$

(iv) $\inf_{x \in X} \{f(x) - h(x)\} = \inf_{x \in X}\{f^{cc'}(x) - h^{cc'}(x)\} \quad \forall\, f$

(v) $\inf_{x \in X} \{f(x) - h(x)\} = \inf_{x \in X}\{f^{cc'}(x) - h(x)\} \quad \forall\, f$

$f : X \longrightarrow \overline{\mathbb{R}}$ is c-subdifferentiable at $x_0 \in X$ if
$f(x_0) \in \mathbb{R}$ and
there exists $y_0 \in Y$ such that $c(x_0, y_0) \in \mathbb{R}$ and

$$f(x) - f(x_0) \geq c(x, y_0) - c(x_0, y_0) \qquad (x \in X).$$

y_0 is a $c-$subgradient of f at x_0.
$\partial_c f(x_0) = \{y_0 \in Y : y_0 \text{ is a } c\text{-subgradient of } f \text{ at } x_0\}$
$\partial_c f(x_0) = \emptyset$ if $f(x_0) \notin \mathbb{R}$

PROPOSITION. Let $f : X \longrightarrow \overline{\mathbb{R}}$, $x_0 \in X$ and $y_0 \in Y$.
If $c(x_0, y_0) \in \mathbb{R}$ then

$$y_0 \in \partial_c f(x_0) \text{ if and only if } f(x_0) + f^c(y_0) = c(x_0, y_0),$$
$$y_0 \in \partial_c f^{cc'}(x_0) \text{ if and only if } x_0 \in \partial_{c'} f^c(y_0),$$
$$\partial_c f(x_0) \neq \emptyset \text{ implies that } f \text{ is } \Phi_c - \text{convex at } x_0,$$
$$f \text{ is } \Phi_c - \text{convex at } x_0 \text{ implies } \partial_c f^{cc'}(x_0) = \partial_c f(x_0).$$

$\rho : X \rightrightarrows Y$ is c-cyclically monotone if
$(c(x_1, y_0) - c(x_0, y_0)) + (c(x_2, y_1) - c(x_1, y_1)) + \cdots +$
$(c(x_0, y_m) - c(x_m, y_m))$
is nonpositive for any set of pairs (x_i, y_i), $i = 0, 1, ..., m$
(m arbitrary) such that $y_i \in \rho(x_i)$.

THEOREM. $\rho : X \rightrightarrows Y$ is c-cyclically monotone
if and only if
there exists a Φ_C-convex function $f : X \to \mathbb{R} \cup \{+\infty\}$ such that $\rho(x) \subseteq \partial_c f(x)$ for every $x \in X$.

EXAMPLE. $X = Y = \{1, 2\}$

$$\begin{aligned} c &: X \times Y \to \mathbb{R} \\ c(i,j) &= \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \end{aligned}$$

$$f_1, f_2, f_3 : X \to \mathbb{R}$$
$$f_1(1) = f_1(2) = 1$$
$$f_2(1) = 1, f_2(2) = 0$$
$$f_3(1) = 1, f_3(2) = \tfrac{3}{2}$$

$$f_1(x) = \max\{c(x,1), c(x,2)\}$$
$$f_2(x) = \max\{c(x,1), c(x,2) - 1\}$$
$$f_3(x) = \max\left\{c(x,1), c(x,2) + \frac{1}{2}\right\}$$

$$\partial_c f_1(x) = \{x\} \qquad (x \in X)$$
$$\partial_c f_2(1) = \{1\},\ \partial_c f_2(2) = Y$$
$$\partial_c f_3(x) = \{x\} \qquad (x \in X)$$

THEOREM. For a mapping $\Delta : \overline{\mathbb{R}}^X \to \overline{\mathbb{R}}^Y$,
the following statements are equivalent:

(i) There exists a coupling function $c : X \times Y \to \overline{\mathbb{R}}$ s.t.

$$\Delta(f) = f^c \qquad (f \in \overline{\mathbb{R}}^X).$$

(ii) One has

$$\Delta(\inf_{i \in I} f_i) = \sup_{i \in I} \Delta(f_i) \qquad (\{f_i\}_{i \in I} \subseteq \overline{\mathbb{R}}^X)$$

and

$$\Delta(f + c) = \Delta(f) - c \qquad (f \in \overline{\mathbb{R}}^X, c \in \mathbb{R}).$$

Moreover, in this case c is uniquely determined by Δ :

$$c(x, y) = \Delta(\delta_{\{x\}})(y) \qquad (x \in X,\ y \in Y),$$

$\delta_{\{x\}}$ denoting the indicator function of $\{x\}$.

GENERALIZED CONVEX DUALITY

X, U arbitrary sets, $\varphi : X \times U \to \overline{\mathbb{R}}$

$$(\mathcal{P}_u) \qquad \text{minimize } \varphi(x, u)$$

$$u_0 \in U$$

$$(\mathcal{P}) \qquad \text{minimize } \varphi(x, u_0)$$

$$\begin{aligned} p &: U \to \overline{\mathbb{R}} \\ p(u) &= \inf_{x \in X} \varphi(x, u) \end{aligned}$$

V arbitrary set, $c : U \times V \to \overline{\mathbb{R}}$

$$(\mathcal{D}) \qquad \text{maximize } c(u_0, v) - p^c(v)$$

PROPOSITION. The optimal value of $(\mathcal{D})$ is $p^{cc'}(u_0)$. If it is finite, the optimal solution set is $\partial_c p^{cc'}(u_0)$.

THEOREM. The optimal value of $(\mathcal{D})$ is not greater than the optimal value of $(\mathcal{P})$.
They coincide if and only if p is Φ_c-convex at u_0.
In this case, if the optimal value is finite
then the optimal solution set to $(\mathcal{D})$ is $\partial_c p(u_0)$.

COROLLARY. If $x \in X$ and $v \in V$ satisfy

$$\varphi(x, u_0) = c(u_0, v) - p^c(v)$$

then they are optimal solutions to $(\mathcal{P})$ and $(\mathcal{D})$, respectively.

$$\varphi_x : U \to \overline{\mathbb{R}}$$
$$\varphi_x(u) = \varphi(x, u)$$

$$\begin{aligned} L &: X \times V \to \overline{\mathbb{R}} \\ L(x, v) &= c(u_0, v) - \varphi_x^c(v) \end{aligned}$$

If φ_x is Φ_c-convex at u_0 for every $x \in X$,

$$\begin{aligned} \sup_{v \in V} L(x, v) &= \sup_{v \in V} \{c(u_0, v) - \varphi_x^c(v)\} = \varphi_x^{cc}(u_0) \\ &= \varphi_x(u_0) = \varphi(x, u_0). \end{aligned}$$

If c does not take the value $+\infty$,

$$\begin{aligned}
\inf_{x\in X} L(x,v) &= \inf_{x\in X} \{c(u_0,v) - \varphi_x^c(v)\} \\
&= \inf_{x\in X} \left\{ c(u_0,v) - \sup_{u\in U} \{c(u,v) - \varphi_x(u)\} \right\} \\
&= c(u_0,v) - \sup_{u\in U} \left\{ c(u,v) - \inf_{x\in X} \varphi(x,u) \right\} \\
&= c(u_0,v) - \sup_{u\in U} \{c(u,v) - p(u)\} \\
&= c(u_0,v) - p^c(v).
\end{aligned}$$

EXAMPLE. $f:\Omega\subseteq\mathbb{R}^n\to\overline{\mathbb{R}}$, $g:\Omega\subseteq\mathbb{R}^n\to\mathbb{R}^m$

$$(\mathcal{P})\qquad \begin{array}{cc}\text{minimize} & f(x)\\ \text{subject to} & g(x)\le 0\end{array}$$

$$\begin{aligned}\varphi &: \mathbb{R}^n\times\mathbb{R}^m\to\overline{\mathbb{R}}\\ \varphi(x,u) &= \begin{cases} f(x) & \text{if } x\in\Omega \text{ and } g(x)+u\le 0\\ +\infty & \text{otherwise}\end{cases}\end{aligned}$$

$$c:\mathbb{R}^m\times(\mathbb{R}_+\times\mathbb{R}^m)\to\mathbb{R}$$
$$c\,(u,(\rho,y)) = -\rho\,\|u-y\|^2$$

$$L:\mathbb{R}^n\times\mathbb{R}_+\times\mathbb{R}^m\to\overline{\mathbb{R}}$$
$$L(x,\rho,y) = f(x)+\rho\sum_{i=1}^m\Big(2y_i\max\{g_i(x),-y_i\}+(\max\{g_i(x),-y_i\})^2\Big)\quad \text{if } x\in\Omega$$
$$= +\infty \qquad \text{if } x\notin\Omega$$

QUASICONVEX CONJUGATION

THEOREM. For any function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$,
the following statements are equivalent:

(i) f is evenly quasiaffine.

(ii) Every nonempty level set of f is either an (open or closed) halfspace or the whole space.

(iii) There exists $x^* \in \mathbb{R}^n$ and a nondecreasing function $h : \mathbb{R} \to \overline{\mathbb{R}}$ such that $f = h \circ \langle \cdot, x^* \rangle$.

X, Y arbitrary sets, $G \subseteq X \times Y$

$$\begin{aligned} c &: X \times Y \to \overline{\mathbb{R}} \\ c(x,y) &= \begin{cases} 0 & \text{if } (x,y) \in G \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

$$\begin{aligned} F &: X \rightrightarrows Y \\ F(x) &= \{y \in Y : (x,y) \in G\} \end{aligned}$$

For $f : X \to \overline{\mathbb{R}}$, $\qquad f^c(y) = -\inf_{x \in F^{-1}(y)} f(x)$

For $g : Y \to \overline{\mathbb{R}}$, $\qquad g^{c'}(x) = -\inf_{y \in F(x)} g(y)$

$$f^{cc'}(x_0) = \sup_{y \in F(x_0)} \inf_{x \in F^{-1}(y)} f(x) \qquad (x_0 \in X)$$

$f : X \to \overline{\mathbb{R}}$, $x_0 \in f^{-1}(\mathbb{R})$, $y_0 \in Y$

$$y_0 \in \partial_c f(x_0) \iff y_0 \in F(x_0),\ f(x_0) = \inf_{x \in F^{-1}(y_0)} f(x)$$

THEOREM. Let $f : X \to \overline{\mathbb{R}}$ and $x_0 \in X$.
Then f is Φ_c-convex at x_0 if and only if
$\forall\ \lambda < f(x_0)\ \exists\ y_\lambda \in F(x_0)$ s.t. $S_\lambda(f) \cap F^{-1}(y_\lambda) = \emptyset$.

COROLLARY. A function $f : X \to \overline{\mathbb{R}}$. is Φ_c-convex
if and only if
each level set of f is of the form $\bigcap_{y \in Y_\lambda} \left(X \setminus F^{-1}(y) \right)$ with
$Y_\lambda \subseteq Y$.

THEOREM. For any $h : X \to \overline{\mathbb{R}}$, the following statements are equivalent:

(i) h is Φ_C-convex.

(ii) $\inf_{x \in X} \max \{f(x), -h(x)\} = \inf_{x \in X} \max \left\{f(x), -h^{cc'}(x)\right\}$ $\forall\ f : X \to \overline{\mathbb{R}}$.

(iii) $\inf_{x \in X} \max \{f(x), -h(x)\} = \inf_{y \in Y} \max \{h^c(y), -f^c(y)\}$ $\forall\ f : X \to \overline{\mathbb{R}}$.

(iv) $\inf_{x \in X} \max \{f(x), -h(x)\} = \inf_{x \in X} \max \left\{f^{cc'}(x), -h^{cc'}(x)\right\}$ $\forall\ f : X \to \overline{\mathbb{R}}$.

(v) $\inf_{x \in X} \max \{f(x), -h(x)\} = \inf_{x \in X} \max \left\{f^{cc'}(x), -h(x)\right\}$ $\forall\ f : X \to \overline{\mathbb{R}}$.

$$X = \mathbb{R}^n, Y = \mathbb{R}^n \times \mathbb{R}$$

$$G = \{(x, x^*, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} : \langle x, x^* \rangle \geq t\}$$

$$\begin{aligned} f &: \mathbb{R}^n \to \overline{\mathbb{R}} \\ f^c(x^*, t) &= -\inf \{f(x) : \langle x, x^* \rangle \geq t\} \end{aligned}$$

$$
\begin{aligned}
g &: \mathbb{R}^n \times \mathbb{R} \to \overline{\mathbb{R}} \\
g^{c'}(x) &= -\inf \{g(x^*, t)) : \langle x, x^* \rangle \geq t\}
\end{aligned}
$$

$$
f^{cc'}(x_0) = \sup_{x^* \in \mathbb{R}^n} \inf \{f(x) : \langle x, x^* \rangle \geq \langle x_0, x^* \rangle\}
$$

THEOREM.A function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is Φ_c-convex if and only if it is evenly quasiconvex.

COROLLARY. The second $c-$conjugate $f^{cc'}$ of any function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ coincides with the evenly quasiconvex hull of f.

COROLLARY. Every evenly quasiconvex function is the pointwise supremum of a collection of evenly quasiaffine functions.

If f is evenly quasiconvex,

$$
f = \sup_{x^* \in \mathbb{R}^n} \varphi_{x^*}, \text{ with } \varphi_{x^*} = \inf \{f(x) : \langle x, x^* \rangle \geq \langle \cdot, x^* \rangle\}.
$$

THEOREM. Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, $x_0 \in \mathbb{R}^n$ be such that $f(x_0) \in \mathbb{R}$, and let

$$\partial^{GP} f(x_0) = \{x^* \in \mathbb{R}^n : \langle x, x^* \rangle < \langle x_0, x^* \rangle \quad \forall\, x \in \mathbb{R}^n \text{ s. t. } f(x) < f(x_0)\}.$$

Then

$$\partial_c f(x_0) = \{(x^*, t) \in \mathbb{R}^n \times \mathbb{R} : x^* \in \partial^{GP} f(x_0), t \leq \langle x_0, x^* \rangle ,\ \inf\{f(x) : \langle x, x^* \rangle \geq t\} = f(x_0)\}.$$

COROLLARY. Let f and x_0 be as above. Then

$$\partial^{GP} f(x_0) = \{x^* \in \mathbb{R}^n : (x^*, \langle x_0, x^* \rangle) \in \partial_c f(x_0)\}\,.$$

COROLLARY. Let f and x_0 be as above. Then $\partial^{GP} f(x_0)$ coincides with the projection of $\partial_c f(x_0)$ onto $\mathbb{R}^n$.

X real vector space, ρ order relation on X

ρ is compatible with the linear structure of X if

$$x_1\ \rho\ y_1 \ ,\ \ x_2\ \rho\ y_2 \quad \Longrightarrow \quad x_1 + x_2\ \rho\ y_1 + y_2$$

and

$$x\ \rho\ y \ ,\ \ \lambda \geq 0 \quad \Longrightarrow \quad \lambda\, x\ \rho\ \lambda y.$$

THEOREM. Let X be a real vector space.
A function $f : X \to \overline{\mathbb{R}}$ is quasiconvex if and only if there is a family $\{\rho_i\}_{i \in I}$ of total order relations on X that are compatible with its linear structure and a corresponding family $\{f_i\}_{i \in I}$ of isotonic functions $f_i : (X, \rho_i) \to (\overline{\mathbb{R}}, \leq)$ such that

$$f(x) = \max_{i \in I} f_i(x) \qquad (x \in X).$$

QUASICONVEX DUALITY

X, U arbitrary sets, $\varphi : X \times U \to \overline{\mathbb{R}}$

$$(\mathcal{P}_u) \qquad \text{minimize } \varphi(x, u)$$

$$\begin{aligned} p \quad &: \quad U \to \overline{\mathbb{R}} \\ p(u) \quad &= \quad \inf_{x \in X} \varphi(x, u) \end{aligned}$$

$$u_0 \in U$$

$$(\mathcal{P}) \qquad \text{minimize } \varphi(x, u_0)$$

V arbitrary set, $G \subseteq U \times V$

$$\begin{aligned} F &: U \rightrightarrows V \\ F(u) &= \{v \in V : (u,v) \in G\} \end{aligned}$$

$$c(u,v) = \begin{cases} 0 & \text{if } (u,v) \in G \\ -\infty & \text{otherwise} \end{cases}$$

$$(\mathcal{D}) \qquad \text{maximize } c(u_0, v) - p^c(v)$$

$$(\mathcal{D}) \qquad \text{maximize } \inf_{x \in X,\ u \in F^{-1}(v)} \varphi(x,u) \\ \text{subject to } v \in F(u_0)$$

$$L : X \times V \to \overline{\mathbb{R}}$$

$$L(x,v) = c(u_0,v) - \varphi_x^c(v) = c(u_0,v) + \inf_{u \in F^{-1}(v)} \varphi(x,v)$$

$$= \begin{cases} \inf_{u \in F^{-1}(v)} \varphi(x,v) & \text{if } v \in F(u_0) \\ -\infty & \text{otherwise} \end{cases}$$

$$\begin{aligned} U &= \mathbb{R}^m,\ V = \mathbb{R}^m \times \mathbb{R}, \\ G &= \{(u,u^*,t) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R} : \langle u, u^* \rangle \geq t\},\ u_0 = 0 \end{aligned}$$

$$(\mathcal{D}) \qquad \text{maximize } \inf_{x \in X,\ \langle u,u^* \rangle \geq t} \varphi(x,u) \\ \text{subject to } t \leq 0$$

$$(\mathcal{D}) \qquad \text{maximize} \quad \inf_{x \in X,\ \langle u, u^* \rangle \geq 0} \varphi(x, u)$$

$$\begin{aligned} L &: \mathbb{X} \times \mathbb{R}^m \times \mathbb{R} \\ L(x, u^*, t) &= \begin{cases} \inf \{\varphi(x, u) : \langle u, u^* \rangle \geq t\} & \text{if } t \leq 0 \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

PROPOSITION. The optimal value of $(\mathcal{D})$ coincides with the value of the evenly quasiconvex hull of p at 0.

THEOREM. The optimal value of $(\mathcal{D})$ is not greater than the optimal value of $(\mathcal{P})$. They coincide if and only if p is evenly quasiconvex at 0.

COROLLARY. Let $x_0 \in X$ be an optimal solution to $(\mathcal{P})$. Then $u^* \in \mathbb{R}^n$ is an optimal solution to $(\mathcal{D})$ if $(x_0, 0) \in X \times \mathbb{R}^m$ minimizes $\varphi(x, u)$ subject to the constraint $\langle u, u^* \rangle \geq 0$.

$$f : \Omega \subseteq \mathbb{R}^n \to \overline{\mathbb{R}},\ g : \Omega \to \mathbb{R}^m$$

$$\varphi(x,u) = \begin{cases} f(x) & \text{if } x \in \Omega,\ g(x) + u \leq 0 \\ +\infty & \text{otherwise} \end{cases}$$

$$\begin{aligned} &\inf_{x \in X,\ \langle u, u^* \rangle \geq 0} \varphi(x,u) \\ &\quad = \inf \{ f(x) : g(x) + u \leq 0,\ \ \langle u, u^* \rangle \geq 0 \} \\ &\quad = \begin{cases} \inf \{ f(x) : \langle g(x), u^* \rangle \leq 0 \} & \text{if } u^* \geq 0 \\ \inf f(\Omega) & \text{if } u^* \not\geq 0 \end{cases} \end{aligned}$$

$$(\mathcal{P}) \qquad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g(x) \leq 0 \end{array}$$

$$(\mathcal{D}) \qquad \begin{array}{ll} \text{maximize} & \inf \{ f(x) : \langle g(x), u^* \rangle \leq 0 \} \\ \text{subject to} & u^* \geq 0 \end{array}$$

$$\begin{aligned} L \ &: \ \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \overline{\mathbb{R}} \\ L(x, u^*, t) \ &= \ \begin{cases} \inf \{ f(x) : \langle g(x), u^* \rangle + t \leq 0 \} & \\ & \text{if } u^* \geq 0 \text{ and } t \leq 0 \\ \inf f(\Omega) & \text{if } u^* \not\geq 0 \text{ and } t \leq 0 \\ -\infty & \text{if } t > 0 \end{cases} \end{aligned}$$

THEOREM. If f is quasiconvex and u.s.c. along lines, the component functions of g are convex and there is an $x \in \Omega$ such that $g(x) < 0$ (componentwise) then

$$\inf \{f(x) : g(x) \leq 0\} = \max_{u^* \geq 0} \inf \{f(x) : \langle g(x), u^* \rangle \leq 0\} .$$

Hence, $u^* \geq 0$ is an optimal solution to $(\mathcal{D})$ if and only if the optimal value of $(\mathcal{P})$ coincides with that of

$$(\mathcal{S}_{u^*}) \qquad \begin{array}{rc} \text{minimize} & f(x) \\ \text{subject to} & \langle g(x), u^* \rangle \leq 0. \end{array}$$

In this case, any optimal solution $x_0 \in \Omega$ to $(\mathcal{P})$ is also an optimal solution to $(\mathcal{S}_{u^*})$.