Generalized Convexity in Economics

J.E. Martínez-Legaz
Universitat Autònoma de Barcelona, Spain

2nd Summer School: Generalized Convexity Analysis
Introduction to Theory and Applications

JULY 19, 2008
KAOHSIUNG - TAIWAN

References:

John, Reinhard: Uses of generalized convexity and generalized monotonicity in economics. Handbook of generalized convexity and generalized monotonicity, 619–666, Nonconvex Optim. Appl., 76, Springer, New York, 2005.

Martínez-Legaz, Juan Enrique: Generalized convex duality and its economic applications. Handbook of generalized convexity and generalized monotonicity, 237–292, Nonconvex Optim. Appl., 76, Springer, New York, 2005.

1 Preference, Utility and Demand

n different types of commodities

$\mathbb{R}_+^n$ the set of commodity bundles

$\succsim$ total preorder on $\mathbb{R}_+^n$:
$x, y \in \mathbb{R}_+^n \Longrightarrow x \succsim y$ or $y \succsim x$
$x \succsim y,\ y \succsim z \Longrightarrow x \succsim z$

$\succsim$ is continuous if
$graph(\succsim) := \left\{(x, y) \in \mathbb{R}_+^n \times \mathbb{R}_+^n : x \succsim y\right\}$ is closed.

PROPOSITION. A binary relation $\succsim$ on $\mathbb{R}_+^n$ is a continuous total preorder if and only if there is a continuous function $u : \mathbb{R}_+^n \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}_+^n$

$$x \succsim y \Longleftrightarrow u(x) \geq u(y).$$

$\succsim$ total preorder on $\mathbb{R}^n_+$

$\succsim$ is convex if

$$x \succsim y \Longrightarrow (1-\lambda)\,x + \lambda y \succsim y \text{ for all } \lambda \in [0,1]\,.$$

$$x \succ y \qquad \Longleftrightarrow \qquad x \succsim y, y \not\succsim x$$

$\succsim$ is strictly convex if

$$x \succsim y, x \neq y \Longrightarrow (1-\lambda)\,x + \lambda y \succ y \text{ for all } \lambda \in (0,1)\,.$$

$u : \mathbb{R}^n_+ \to \mathbb{R}$ utility re presentation of $\succsim$

$\succsim$ is convex $\Longleftrightarrow$ u is quasiconcave

$\succsim$ is strictly convex $\Longleftrightarrow$ u is strictly quasiconcave

The demand correspondence: $X = X_u : \mathbb{R}^n_{++} \rightrightarrows \mathbb{R}^n_+$

$$X(p) = \{x \in \mathbb{R}^n_+ : \langle x, p\rangle \leq 1,\ u\,(x) \geq u\,(y) \text{ for all } y \in \mathbb{R}^n_+ \text{ such that } \langle y, p\rangle \leq 1\}.$$

$\succsim$ is said to be locally nonsatiated if
no relatively open subset of $\mathbb{R}^n_+$ has a $\succsim$-maximal element.

PROPOSITION.
(1) If $\succsim$ is locally nonsatiated then,
for every $p \in \mathbb{R}^n_{++}$ and $x \in X(p)$, $\langle x, p \rangle = 1$.

(2) If $\succsim$ is convex then X is convex-valued and satisfies GARP:
if $p^i \in \mathbb{R}^n_{++}$, $x^i \in X(p^i)$ $\quad (i = 1, ..., k)$ then

$$\min_{i=1,\dots,k-1} \left\langle x^i - x^{i+1}, p^i \right\rangle \geq 0 \Longrightarrow \left\langle x^k - x^1, p^k \right\rangle \leq 0.$$

(3) If u is u.s.c. and strictly quasiconcave then X is single-valued and satisfies SARP:
if $p^i \in \mathbb{R}^n_{++}$, $x^i \in X(p^i)$ $\quad (i = 1, ..., k)$ then

$$\min_{i=1,\dots,k-1} \left\langle x^i - x^{i+1}, p^i \right\rangle \geq 0 \Longrightarrow \left\langle x^k - x^1, p^k \right\rangle < 0.$$

2 Duality in Consumer Theory

$u : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$ utility function

$p \in \mathbb{R}^n_+$ price vector

$M > 0$ income

$$(\mathcal{P}) \qquad \begin{array}{ll} \text{maximize} & u(x) \\ \text{subject to} & \langle x, p \rangle \le M \end{array}$$

The indirect utility function $v : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$:

$$v(p) = \sup \{u(x) : \langle x, p \rangle \le 1\}$$

$$S_\lambda(v) = \bigcap_{x : u(x) > \lambda} \left\{ p \in \mathbb{R}^n_+ : \langle x, p \rangle > 1 \right\} \qquad (\lambda \in \mathbb{R})$$

THEOREM. Let $v : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$. There exists a utility function $u : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$ having v as its associated indirect utility function if and only if v is nonincreasing, evenly quasiconvex and satisfies

$$v(p) \leq \lim_{\alpha \to 1^-} \overline{v}(\alpha p) \qquad (p \in bd\, \mathbb{R}^n_+),$$

$\overline{v}$ denoting the l.s.c. hull of v.

In this case one can take u nondecreasing, evenly quasi-concave and satisfying

$$u(x) \geq \lim_{\alpha \to 1^-} \underline{u}(\alpha x) \qquad (x \in bd\, \mathbb{R}^n_+). \qquad (1)$$

Under these conditions u is unique, namely, u is the pointwise largest utility function inducing v; furthermore, it satisfies

$$u(x) = \inf \{v(p) : \langle x, p \rangle \leq 1\} \qquad (x \in \mathbb{R}^n_+). \qquad (2)$$

COROLLARY. For every $v : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$, the function $v_0 : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$ defined by

$$v_0(p) = \sup \{u(x) : \langle x, p \rangle \leq 1\},$$

with u given by (2), is the pointwise largest nonincreasing evenly quasiconvex minorant of v that satisfies

$$v_0(p) \leq \lim_{\alpha \to 1^-} \overline{v_0}(\alpha p) \qquad (p \in bd\ \mathbb{R}^n_+). \qquad (3)$$

THEOREM. Let $u : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$. There exists a function $v : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$ such that (2) holds if and only if u is nondecreasing, evenly quasiconcave and satisfies (1).

In this case one can take v nonincreasing, evenly quasiconvex and satisfying (3). Under these conditions v is unique, namely, v is the pointwise smallest function such that (2) holds; furthermore, it is the indirect utility function associated with u.

COROLLARY. For every $u : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$, the function $u^0 : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$ defined by

$$u^0(x) = \inf \{v(p) : \langle x, p \rangle \leq 1\},$$

v being the indirect utility function associated with u, is the pointwise smallest nondecreasing evenly quasiconcave majorant of u that satisfies

$$u^0(x) \geq \underline{\lim}_{\alpha \to 1^-} u^0(\alpha x) \qquad (x \in bd\ \mathbb{R}^n_+).$$

THEOREM. A nondecreasing evenly quasiconcave function $u : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$ satisfying (1) is finite-valued if and only if its associated indirect utility function $v : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$ is bounded from below and finite-valued on the interior of $\mathbb{R}^n_+$.

THEOREM. The indirect utility function $v : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$ induced by a nondecreasing evenly quasiconcave function $u : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$ satisfying (1) is finite-valued if and only if u is bounded from above and finite-valued on the interior of $\mathbb{R}^n_+$.

COROLLARY. Let $u : \mathbb{R}^n_+ \longrightarrow \overline{\mathbb{R}}$ be a nondecreasing evenly quasiconcave function satisfying (1) and let $v : \mathbb{R}^n_+ \longrightarrow \overline{\mathbb{R}}$ be its associated indirect utility function. The following statements are equivalent:

(i) u and v are finite-valued.

(ii) u is bounded.

(iii) v is bounded.

$u : \mathbb{R}^n_+ \longrightarrow \mathbb{R}$ utility function

The expenditure function: $e_u : \mathbb{R}^n_+ \times \mathbb{R} \longrightarrow \mathbb{R}_+ \cup \{+\infty\}$

$$e_u(p, \lambda) = \inf \{\langle x, p\rangle : u(x) \geq \lambda\} \qquad \Big((p, \lambda) \in \mathbb{R}^n_+ \times \mathbb{R}\Big).$$

THEOREM. A function $e : \mathbb{R}^n_+ \times \mathbb{R} \to \mathbb{R}_+ \cup \{+\infty\}$ is the expenditure function e_u for some utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ if and only if the following conditions hold:

(i) For each $\lambda \in \mathbb{R}$, either $e(\cdot, \lambda)$ is finite-valued, concave, linearly homogeneous and u.s.c., or it is identically equal to $+\infty$.

(ii) For each $p \in \mathbb{R}^n_+$, $e(p, \cdot)$ is nondecreasing.

(iii) $\bigcup_{\lambda \in \mathbb{R}} \partial e(\cdot, \lambda)(0) = \mathbb{R}^n_+$, with $\partial e(\cdot, \lambda)(0)$ denoting the superdifferential of the concave function $e(\cdot, \lambda)$ at the origin, i.e.

$$\partial e(\cdot, \lambda)(0) = \left\{ x \in \mathbb{R}^n_+ : \langle x, p \rangle \geq e(p, \lambda) \qquad \forall\, p \in \mathbb{R}^n_+ \right\}.$$

THEOREM. The mapping $u \longmapsto e_u$ is a bijection from the set of u.s.c. nondecreasing quasiconcave functions $u : \mathbb{R}^n_+ \to \mathbb{R}$ onto the set of functions $e : \mathbb{R}^n_+ \times \mathbb{R} \to \mathbb{R}_+ \cup \{+\infty\}$ that satisfy *(i)-(iii)*,

(iv) $\bigcap_{\lambda \in \mathbb{R}} \partial e(\cdot, \lambda)(0) = \emptyset$

and

(v) $\bigcap_{\mu < \lambda} \partial e(\cdot, \mu)(0) = \partial e(\cdot, \lambda)(0) \qquad (\lambda \in \mathbb{R})\,.$

Furthermore, the inverse mapping is $e \longmapsto u_e$, with $u_e : \mathbb{R}^n_+ \to \mathbb{R}$ given by

$$u_e(x) = \sup \{\lambda \in \mathbb{R} : x \in \partial e(\cdot, \lambda)(0)\}\,.$$

3 Monotonicity of Demand Functions

The demand correspondence: $X : \mathbb{R}^n_{++} \rightrightarrows \mathbb{R}^n_+$

$$X(p) = \left\{ x \in \mathbb{R}^n_+ : \langle x, p \rangle \le 1,\ u(x) = v(p) \right\}$$

THEOREM. If the utility function $u : \mathbb{R}^n_+ \to \overline{\mathbb{R}}$ has no local maximum then the demand function
$X : \mathbb{R}^n_{++} \rightrightarrows \mathbb{R}^n_+$ is cyclically quasimonotone (in the decreasing sense), i.e.
if $p^i \in \mathbb{R}^n_{++},\ x^i \in X(p^i) \qquad (i = 1, ..., k)$
then

$$\min_{i=1,\dots,k} \left\langle x^i - x^{i+1}, p^i \right\rangle \le 0, \qquad \text{with } x^{k+1} = x^1.$$

X is quasimonotone (in the decreasing sense) if

$$\begin{aligned} p, q \quad &\in \quad \mathbb{R}^n_{++},\ x \in X(p),\ y \in Y(q) \\ &\Longrightarrow \quad \min \{ \langle x - y, p \rangle , \langle y - x, q \rangle \} \le 0. \end{aligned}$$

X is monotone (in the decreasing sense) if

$$\begin{aligned} p, q \quad &\in \quad \mathbb{R}^n_{++},\ x \in X(p),\ y \in Y(q) \\ &\implies \quad \langle x - y, p \rangle + \langle y - x, q \rangle \leq 0. \end{aligned} \tag{4}$$

$u : \mathbb{R}^2_+ \to \mathbb{R}$

$$u(x_1, x_2) = \begin{cases} x_1 + x_2 - 1 & \text{if } x_1 + x_2 < 1 \\ x_1 & \text{if } x_1 + x_2 \geq 1 \end{cases}$$

$$\begin{array}{ll} \text{maximize} & u(x_1, x_2) \\ \text{subject to} & p_1 x_1 + p_2 x_2 \leq 1 \end{array}$$

For $p_1 \geq 1$ and $1 \geq p_2 > 0$, the optimal solution is

$$x_1 = \frac{1 - p_2}{p_1 - p_2}, \qquad x_2 = \frac{p_1 - 1}{p_1 - p_2}.$$

Let $\overline{p} \in \mathbb{R}^n_{++}$ and $\overline{x} \in X(\overline{p})$.

$\overline{p}$ is an optimal solution to

$$\begin{array}{ll} \text{minimize} & v(p) \\ \text{subject to} & \langle \overline{x}, p \rangle \leq 1. \end{array}$$

There exists $\lambda \geq 0$ such that

$$\nabla v(\overline{p}) + \lambda \overline{x} = 0 \qquad \text{and} \qquad \lambda \left(\langle \overline{x}, \overline{p} \rangle - 1 \right) = 0.$$

$$\langle \nabla v(\overline{p}), \overline{p} \rangle + \lambda \langle \overline{x}, \overline{p} \rangle = 0 \qquad \lambda \langle \overline{x}, \overline{p} \rangle = \lambda$$

$$\lambda = - \langle \nabla v(\overline{p}), \overline{p} \rangle$$

$$\nabla v(\overline{p}) - \langle \nabla v(\overline{p}), \overline{p} \rangle \, \overline{x} = 0.$$

THEOREM. (Roy's Identity) If the utility function u is u.s.c. and its associated indirect utility function v is continuously differentiable at $\overline{p} \in \mathbb{R}^n_{++}$, with $\nabla v(\overline{p}) \neq 0$, then

$$X(\overline{p}) = \left\{ \frac{1}{\langle \nabla v(\overline{p}), \overline{p} \rangle} \nabla v(\overline{p}) \right\}.$$

THEOREM. If the utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ is concave, C^2, has a componentwise strictly positive gradient on $\mathbb{R}^n_{++}$, induces a demand function $\varphi : \mathbb{R}^n_{++} \to \mathbb{R}^n_+$ (i.e. a single-valued demand correspondence $p \in \mathbb{R}^n_{++} \rightrightarrows X(p) = \{\varphi(p)\}$), with φ of class C^1, and satisfies

$$-\frac{\left\langle x, \nabla^2 u(x) x \right\rangle}{\langle x, \nabla u(x) \rangle} < 4 \qquad \left(x \in \mathbb{R}^n_{++} \right)$$

then φ is strictly monotone, i.e. it satisfies (4) as a strict inequality whenever $p \neq q$.

$$X(p) = \left\{x \in \mathbb{R}^n_+ : u(x) = v(p)\right\} \qquad \left(p \in \mathbb{R}^n_{++}\right)$$

$$X^{-1}(x) = \left\{p \in \mathbb{R}^n_{++} : -v(p) = -u(x)\right\} \qquad \left(x \in \mathbb{R}^n_+\right)$$

$$-u(x) = \sup\{-v(p) : \langle x, p\rangle \le 1\} \qquad (x \in \mathbb{R}^n_+)$$

PROPOSITION. For a C^2 nondecreasing quasiconcave utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ with no stationary points, the following statements are equivalent:

(i) $-\frac{\langle x, \nabla^2 u(x) x\rangle}{\langle x, \nabla u(x)\rangle} \le 4 \qquad \left(x \in \mathbb{R}^n_{++}\right).$

(ii) The function $(x_1, ..., x_n) \in \mathbb{R}^n_{++} \longmapsto u(x_1^{-\frac{1}{3}}, ..., x_n^{-\frac{1}{3}})$ is convex-along-rays.

(iii) The restriction $v : \mathbb{R}^n_{++} \to \mathbb{R} \cup \{+\infty\}$ of the indirect utility function to the positive orthant has a representation of the type

$$v(p) = \max_{(y,c)\in U} \left\{c - (\langle y, p\rangle)^3\right\} \qquad (p \in \mathbb{R}^n_{++}),$$

with $U \subseteq \left(\mathbb{R}^n_{++} \cup \{0\}\right) \times \mathbb{R}$.

THEOREM. Let $\varphi : \mathbb{R}^n_{++} \to \mathbb{R}^n_+$ be a C^1 demand function induced by a strictly quasiconcave utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$, which is C^2 on $\mathbb{R}^n_{++}$ and has a componentwise strictly positive gradient on $\mathbb{R}^n_{++} \cup \varphi\left(\mathbb{R}^n_{++}\right)$. Then φ is monotone if and only if

$$-\frac{\left\langle x, \nabla^2 u(x) x \right\rangle}{\langle x, \nabla u(x) \rangle} \leq 4 - \frac{\langle x, \nabla u(x) \rangle}{\left\langle \nabla u(x), \left(\nabla^2 u(x)\right)^{-1} \nabla u(x) \right\rangle}$$

$$\forall\, x \in \mathbb{R}^n_{++} \text{ such that } \nabla^2 u(x) \text{ is nonsingular}$$

and

$$-\frac{\left\langle x, \nabla^2 u(x) x \right\rangle}{\langle x, \nabla u(x) \rangle} \leq 4 \qquad \forall\, x \in \mathbb{R}^n_{++} \text{ s. t. } \nabla^2 u(x) \text{ is singular.}$$

PROPOSITION. Let $u : \mathbb{R}^n_+ \to \mathbb{R}$ be a utility function and let $v : \mathbb{R}^n_+ \to \mathbb{R} \cup \{+\infty\}$ be its associated indirect utility function. If the set

$$\Big\{(p, x) \in \mathbb{R}^n_{++} \times \mathbb{R}^n_+ : u(x) - v(p) \geq 0\Big\}$$

is convex (in particular, if the function

$$\psi : \mathbb{R}^n_{++} \times \mathbb{R}^n_+ \to \mathbb{R} \cup \{+\infty\}$$

defined by

$$\psi(p, x) = u(x) - v(p)$$

is quasiconcave) and u has no maximum then the demand correspondence X is monotone.

COROLLARY. Let $u : \mathbb{R}^n_+ \to \mathbb{R}$ be a utility function and let $v : \mathbb{R}^n_+ \to \mathbb{R} \cup \{+\infty\}$ be its associated indirect utility function. If u is concave and has no maximum and v is convex then the demand correspondence X is monotone.

THEOREM. If $u : \mathbb{R}^n_+ \to \mathbb{R}$ is nondecreasing and the function $x \in \mathbb{R}^n_{++} \longmapsto u(x^{-1})$ is convex-along-rays then the restriction $v : \mathbb{R}^n_{++} \to \mathbb{R} \cup \{+\infty\}$ of the indirect utility function to the positive orthant is convex. Conversely, if $v : \mathbb{R}^n_{++} \to \mathbb{R}$ is bounded, nonincreasing and convex then there is a nondecreasing quasiconcave utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ such that $x \in \mathbb{R}^n_{++} \longmapsto u(x^{-1})$ is convex-along-rays and whose associated indirect utility function extends v.

COROLLARY. Let $u : \mathbb{R}^n_+ \to \mathbb{R}$ be a C^2 nondecreasing quasiconcave utility function. The restriction
$v : \mathbb{R}^n_{++} \to \mathbb{R}$ of its associated indirect utility function to the positive orthant is convex if and only if

$$< x, \nabla^2 u(x) x > +2 < \nabla u(x), x > \geq 0 \qquad (x \in \mathbb{R}^n_{++}).$$

4 Consumer Theory without Utility

Let $\succsim$ be a total preorder on $\mathbb{R}^n_+$.

For $x_1, x_2 \in \mathbb{R}^n_+$,

$$\begin{array}{rclcl} x_1 & \sim & x_2 & \Longleftrightarrow & x_1 \succsim x_2,\ x_2 \succsim x_1 \\ x_1 & \succ & x_2 & \Longleftrightarrow & x_1 \succsim x_2,\ x_2 \not\succsim x_1 \end{array}$$

The indirect preorder induced by $\succsim$: For $p_,, p_2 \in \mathbb{R}^n_+$,

$$p_1 \succsim^i p_2 \iff \forall\ x_2 \in B(p_2) = \left\{ x \in \mathbb{R}^n_+ : \langle x, p_2 \rangle \leq 1 \right\} \\ \exists\ x_1 \in B(p_1) \text{ s. t. } x_1 \succsim x_2.$$

$\succsim^i$ is a total preorder on $\mathbb{R}^n_+$.

$\backsim^i$ the associated indifferent relation

$\succ^i$ the associated strict preorder

Let $\succsim^*$ be a total preorder on $\mathbb{R}^n_+$.

The direct preorder induced by $\succsim^*$: For $p_1, p_2 \in \mathbb{R}^n_+$,

$$x_1 \succsim^{*d} x_2 \iff \forall\, p_1 \in B^{-1}(x_1) = \left\{p \in \mathbb{R}^n_+ : \langle x_1, p\rangle \le 1\right\}$$
$$\exists\, p_2 \in B^{-1}(x_2) \text{ s. t. } p_1 \succsim^* p_2.$$

$\succsim^{*d}$ is a total preorder on $\mathbb{R}^n_+$.

THEOREM. Let $\succsim$ be a total preorder on $\mathbb{R}^n_+$. The following statements are equivalent:

(i) $\succsim$ coincides with $\succsim^{id}$.

(ii) $\succsim$ has the following properties:

(a) $\succsim$ is nondecreasing.

(b) For every $x_1 \in \mathbb{R}^n_+$,
the set $\left\{x \in \mathbb{R}^n_+ : x \succsim x_1\right\}$ is evenly convex.

(c) For every $x_1 \in \mathbb{R}^n_+$,
if $x_2 \in cl\ \left\{x \in \mathbb{R}^n_+ : x \succsim x_1\right\}$ and $\alpha > 1$ then $\alpha x_1 \succsim x_2$.

(d) For every $x_1, x_2 \in \mathbb{R}^n_+$,
if $x_1 \sim x_2$ and x_1 is a $\succsim$-maximal element of $B(p_1)$ for some $p_1 \in \mathbb{R}^n_+$ then x_2 is a $\succsim$-maximal element of $B(p_2)$ for some $p_2 \in \mathbb{R}^n_+$.

Moreover, if conditions *(a)*-*(d)* hold, then $\succsim^i$ has the following properties:

(a') $\succsim^i$ is nonincreasing.

(b') For every $p_1 \in \mathbb{R}^n_+$,
the set $\left\{p \in \mathbb{R}^n_+ : p_1 \succsim^i p\right\}$ is evenly convex.

(c*')* For every $p_1 \in \mathbb{R}^n_+$,
if $p_2 \in cl\ \left\{p \in \mathbb{R}^n_+ : p_1 \succsim^i p\right\}$ and $\alpha > 1$ then $p_1 \succsim^i p_2$.

(d') For every $p_1, p_2 \in \mathbb{R}^n_+$,
if $p_1 \backsim^i p_2$ and p_1 is a $\succsim^i -$ minimal element of $B^{-1}(x_1)$ for some $x_1 \in \mathbb{R}^n_+$ then p_2 is a $\succsim^i -$ minimal element of $B^{-1}(x_2)$ for some $x_2 \in \mathbb{R}^n_+$.

The duality mapping $\succsim \mapsto \succsim^i$ is a bijection, with inverse $\succeq^* \mapsto \succeq^{*d}$, from the set of all total preorders $\succeq$ on $\mathbb{R}^n_+$ with properties *(a)*-*(d)* onto the set of all total preorders $\succsim^i$ on $\mathbb{R}^n_+$ with properties *(a')*-*(d')*.

$u : \mathbb{R}^2_+ \longrightarrow \mathbb{R}$

$$u(y,z) = \begin{cases} -\frac{1}{y+1} & \text{if } z \leq 1 \\ 1 & \text{if } z > 1 \end{cases}$$

$v : \mathbb{R}^2_+ \longrightarrow \mathbb{R}$

$$v(q,r) = \begin{cases} 1 & \text{if } r < 1 \\ 0 & \text{if } r \geq 1 \text{ and } q = 0 \\ -\frac{q}{q+1} & \text{if } r \geq 1 \text{ and } q > 0 \end{cases} .$$

If $v(q,r) = 0$ then no $\succeq -$ maximal element exists in $B(q,r)$.

If $v(q,r) \neq 0$ then $B(q,r)$ has at least a maximal element.

v is a utility representation for $\succsim^i$.

$(0,1) \backsim^i (0,2)$

$(0,1)$ is a $\succsim^i -$ minimal element of $B^{-1}(1,1)$

No $(y,z) \in \mathbb{R}^2_+$ exists such that $(0,2)$ is a $\succsim^i -$ minimal element of $B^{-1}(y,z)$:

If $(y,z) \in \mathbb{R}^2_+$, with $y \neq 0$, is such that $(0,2) \in B^{-1}(y,z)$ then $(0,2) \succ^i \left(\frac{2-3z}{2y}, \frac{3}{2}\right) \in B^{-1}(y,z)$;

if $(0,z) \in \mathbb{R}^2_+$ is such that $(0,2) \in B^{-1}(0,z)$ then $(0,2) \succ^i \left(\frac{3}{2}, \frac{3}{2}\right) \in B^{-1}(0,z)$.

$\succsim^i \neq \succsim^{idi}$

$u^0 : \mathbb{R}^2_+ \longrightarrow \mathbb{R}$

$$u^0(y,z) = \begin{cases} -1 & \text{if } z \leq 1 \text{ and } y = 0 \\ \frac{z-1}{y-z+1} & \text{if } z \leq 1 \text{ and } y > 0 \\ 1 & \text{if } z > 1 \end{cases}$$

u^0 is a utility representation of $\succsim^{id}$.

For every $(y,z) \in B(0,2)$, $u^0(y,z) < 0 = u^0(1,1)$

$$(1,1) \in B(0,1)$$

$$(0,1) \succ^{idi} (0,2)$$

$$(0,1) \backsim^{i} (0,2)$$

PROPOSITION. Let $\succsim$ be a total preorder on $\mathbb{R}^n_+$ satisfying properties *(a)-(c)*. Then $\succsim^{id}$ is the total preorder whose strict preorder $\succ^{id}$ is defined as follows:

$x_1 \succ^{id} x_2$ if and only if
either $x_1 \succ x_2$
or $x_1 \backsim x_2$, x_1 is not a $\succsim$ $-$maximal element in $B(p)$ for any $p \in \mathbb{R}^n_+$ and x_2 is a $\succsim$-maximal element of $B(p)$ for some $p \in \mathbb{R}^n_+$.

$\succ^{id}$ is an extension of $\succ$.

$\succsim$ is an extension of $\succsim^{id}$.

THEOREM. For every total preorder $\succsim$ on $\mathbb{R}^n_+$, $\succsim^{idid}$ coincides with $\succsim^{id}$.

THEOREM. Let $\succsim$ be a total preorder on $\mathbb{R}^n_+$ such that for every $p \in \mathbb{R}^n_{++}$ the set $B(p)$ has a $\succsim$ $-$maximal element. Then $\succsim^i$ has the following properties:

(a) $\succsim^i$ is nonincreasing.

(b) For every $p_1 \in \mathbb{R}^n_+$, the set $\left\{p \in \mathbb{R}^n_+ : p_1 \succ^i p\right\}$ is evenly convex.

(c) For every $p_1 \in \mathbb{R}^n_+$, if $p_2 \in cl\ \left\{p \in \mathbb{R}^n_+ : p_1 \succ^i p\right\}$ and $\alpha > 1$ then $p_1 \succsim \alpha p_2$.

Conversely, if $\succsim^*$ is a total preorder on $\mathbb{R}^n_+$ such that *(a)*-*(c)* hold with $\succsim^i$ replaced by $\succsim^*$ then, for every $p \in \mathbb{R}^n_{++}$, the set $B(p)$ has a $\succsim^{*d}$ $-$maximal element and $\succsim^{*di}$ coincides with $\succsim^*$.

THEOREM. Let $\succsim$ be a total preorder on $\mathbb{R}^n_+$ that coincides with $\succsim^{id}$. The following statements are equivalent:

(i) For every $p_1 \in \mathbb{R}^n_{++}$ the set $B(p_1)$ has a $\succsim$ $-$maximal element.

(ii) For every $p_1 \in \mathbb{R}^n_+$, the set $\left\{p \in \mathbb{R}^n_+ : p_1 \succ^i p\right\}$ is evenly convex.

(iii) For every $p_1 \in \mathbb{R}^n_{++}$, the set $\left\{p \in \mathbb{R}^n_+ : p_1 \succ^i p\right\}$ is evenly convex.

$\succsim$ partial preorder on $\mathbb{R}^n_+$

The expenditure function: $\qquad e_{\succsim} : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}_+$

$$e_{\succsim}(p, x) = \inf\left\{\left\langle x', p\right\rangle : x' \succsim x\right\} \qquad \left(p \in \mathbb{R}^n_+,\ x \in \mathbb{R}^n_+\right).$$

(HCP) For all $x_1, x_2 \in \mathbb{R}^n_+$,

$$\overline{co}\left(S^{x_1}_{\succsim} + \mathbb{R}^n_+\right) \subseteq \overline{co}\left(S^{x_2}_{\succsim} + \mathbb{R}^n_+\right) \qquad \Longrightarrow \qquad S^{x_1}_{\succsim} \subseteq S^{x_2}_{\succsim}$$

THEOREM. The mapping $\succsim \;\longmapsto\; e_{\succeq}$ is a bijection from the set of all preorders $\succsim$ on $\mathbb{R}_+^n$ whose upper contour sets $S_{\succsim}^x = \left\{x' \in \mathbb{R}_+^n : x' \succsim x\right\}$ satisfy *(HCP)* onto the set of functions $e : \mathbb{R}_+^n \times \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ that satisfy the following properties:

(a) For every $x \in \mathbb{R}_+^n$, the mapping $e(\cdot, x)$ is concave, positively homogeneous and upper semicontinuous.

(b) For every $x \in \mathbb{R}_+^n$, the closed convex hull of the set $\left\{x' \in \mathbb{R}_+^n : e(\cdot, x') \geq e(\cdot, x)\right\} + \mathbb{R}_+^n$ coincides with $\partial e(\cdot, x)(0)$, the superdifferential of the concave function $e(\cdot, x)$ at the origin.

The inverse mapping is $e \longmapsto \succsim_e$, with $\succsim_e$ denoting the preorder on $\mathbb{R}_+^n$ defined by $x_1 \succsim_e x_2$ if and only if $e(\cdot, x_1) \geq e(\cdot, x_2)$ (pointwise).

THEOREM. The mapping $\succsim \;\longmapsto\; e_{\succsim}$ is a bijection from the set of all nondecreasing preorders $\succsim$ on $\mathbb{R}^n_+$ whose upper contour sets are closed and convex onto the set of functions $e : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}_+$ that satisfy the following properties:

(a) For every $x \in \mathbb{R}^n_+$, the mapping $e(\cdot, x)$ is concave, positively homogeneous and upper semicontinuous.

*(b')*For every $x \in \mathbb{R}^n_+$,

$$\Big\{x' \in \mathbb{R}^n_+ : e(\cdot, x') \geq e(\cdot, x)\Big\} = \partial e(\cdot, x)(0).$$

The inverse mapping $e \longmapsto \succsim_e$ is given by: $x_1 \succsim_e x_2$ if and only if $x_1 \in \partial e(\cdot, x_2)(0)$.

$\succsim$ total preorder on $\mathbb{R}^n_+$

The demand correspondence: $\quad X : \mathbb{R}^n_{++} \rightrightarrows \mathbb{R}^n_+$

$$X(p) = \{x \in B(p) : x \succsim y \quad \forall\, y \in B(p)\}$$

$\succsim$ on $\mathbb{R}^n_+$ is said to be locally nonsatiated if no relatively open subset of $\mathbb{R}^n_+$ has a $\succsim$-maximal element.

PROPOSITION. Let $\succsim$ be a locally nonsatiated total preorder in $\mathbb{R}^n_+$ and let X be its associated demand correspondence. If the set

$$\left\{(x,p) \in \mathbb{R}^n_+ \times \mathbb{R}^n_{++} : x \succsim y \quad \forall\, y \in B(p)\right\}$$

is convex then X is monotone.

PROPOSITION. Let $\succsim$ be a nondecreasing total preorder in $\mathbb{R}^n_+$, X be its associated demand correspondence, and assume that $\succsim$ satisfies the following condition:

$$\left.\begin{array}{c} x_1 \succsim \lambda y \\ x_2 \succsim \mu y \\ \lambda > 0,\ \mu > 0 \end{array}\right\} \Longrightarrow \frac{1}{2}(x_1 + x_2) \succsim 2\frac{\lambda\mu}{\lambda+\mu} y.$$

Then X is monotone.

If $graph(\succsim) = \left\{(x,y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ : x \succsim y\right\}$ is convex set then one has

$$\left.\begin{array}{c} x_1 \succsim \lambda y \\ x_2 \succsim \mu y \\ \lambda > 0,\ \mu > 0 \end{array}\right\} \Longrightarrow \tfrac{1}{2}(x_1 + x_2) \succsim \tfrac{\lambda+\mu}{2} y \geq 2\tfrac{\lambda\mu}{\lambda+\mu} y.$$