# Geodesic Convexity and Optimization

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Dedicated to the Memory of My Friend,
Professor Tamás Rapcsák.

Summer School on Generalized Convex Analysis
Kaohsiung, Taiwan, July 15–19, 2008

# Professor Tamás Rapcsák

# Professor Tamás Rapcsák

# Krisztina and Tamás in Varese

# Krisztina, Tamás ond many others in Varese

# Family data, Education, Degrees

Born: March 18, 1947, Debrecen.

Died: March 24, 2008, Cuba.

He was married to Krisztina, they had two children.

Education, Degrees:

- MSc in Mathematics, Lajos Kossuth University of Sciences, Debrecen, 1965-70
- PhD in Operations Research, Lajos Kossuth University of Sciences, Debrecen, 1974
- Candidate of Science, Hungarian Academy of Sciences, 1985
- Habilitation in Applied Mathematics, Operations Research, Technical University of Budapest, 1995
- Doctor of Science, Operations Research and Decision Systems, Hungarian Academy of Sciences, 1998

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# Positions

- 1970—1976: Computer and Automation Research Institute, Hungarian Academy of Sciences (MTA SZTAKI), Department of Operations Research, Researcher
- 1976: Électricité de France, Paris, Scholarship Fellow
- 1976—1978: MTA SZTAKI, Department of Operations Research, Researcher
- 1978—1980: Computer Centre of the Ministry for the Management of Water Supplies, Alger, Algeria, Expert
- 1980—1989: MTA SZTAKI, Department of Operations Research, Senior Researcher
- 1989—1990: MTA SZTAKI, Department of Operations Research, Head of Department of Operations Research, Senior Researcher
- 1991—2008: MTA SZTAKI, Laboratory and Department of Operations Research and Decision Systems, Head of Laboratory and Department
- 1995—2008: Corvinus University of Budapest, Full Professor and Head of Department of Decisions in Economy

# Committees, Societies

- Vice-president, Hungarian Operational Research Society, 1991-1994
- President, Hungarian Operational Research Society, 1994-1996
- Member, Committee for Operations Research, Hungarian Academy of Sciences, 1991-2008
- Representative of the Hungarian Operational Research Society in European Operational Research Society, 1991-2008
- Member, Mathematical Programming Society
- Vice-President, Committee for Operations Research, Hungarian Academy of Sciences, 1996-1999
- Member, Committee for Mathematics at Higher Education, Accreditation Committee, 1997-2000
- President, Hungarian Operational Research Society, 1998-2000
- Member, Board for Hungarian Operatinal Research Society, 2006-2008

# Editorial Boards

- Journal of Optimization Theory and Applications (JOTA)
- Journal of Global Optimization (JoGO)
- Central European Journal for Operations Research (CEJOR)
- Pure Mathematics and Applications (PuMA)
- Alkalmazott Matematikai Lapok (Applied Mathematics Letters)
- Journal of ICT
- Optimization Letters

# Awards

- 1978: Gyula Farkas Prize
- 1978, 1986, 1992, 1996, 1999, 2001: Institute Award, MTA SZTAKI
- 1996: ANBAR Citation of Highest Quality Rating
- 1997, 2000, 2006: Institute Publication Award, MTA SZTAKI
- 1999-2002: Széchenyi Professor Fellowship
- 2003: Gold Medal of Corvinus University of Budapest
- 2003: Bolyai Farkas Prize, Hungarian Academy of Sciences

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- T. Rapcsák, *On the arcwise convexity*, Alkalmaz. Mat. Lapok **10** (1984), no. 1-2, 115–123.
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# Geodesically Convex Sets

Let $M \subset \mathbb{R}^n$ denote a $k$-dimensional Riemannian $C^m$-manifold.

## Definition

A curve $\gamma : [0, 1] \to M$ is called geodesic if its tangent is parallel along the curve. (Equivalently, the curve is the shortest path between any two close points of the curve.)

## Definition

A set $A \subset M$ is called g-convex if any two points of $A$ are joined by a geodesic belonging to $A$, i.e., for all $x, y \in A$, there exists a geodesic curve $\gamma : [0, 1] \to A$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

## Examples

1. A connected, complete Riemannian manifold is always g-convex.
2. For every point $p \in M$, there is a neighborhood $U$ of $p$ which is g-convex; for any two points in $U$, there is a unique geodesic curve joining the two points and staying in $U$.

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## Definition

Let $A \subset M$ be a g-convex set. A function $f : A \to \mathbb{R}$ is called g-convex if, for any two points $x, y \in A$ and arc-length-parametrized geodesic curve $\gamma : [0, \ell] \to A$ with $\gamma(0) = x$ and $\gamma(\ell) = y$, we have

$$f(\gamma(t\ell)) \le tf(\gamma(0)) + (1 - t)f(\gamma(\ell)) \qquad (t \in [0, 1]).$$

## Lemma

Let $A \subset M$ be a g-convex set and $f : A \to \mathbb{R}$ be g-convex function. Then, for all $c \in \mathbb{R}$, the level set

$$\{x \in A \mid f(x) \le c\}$$

is also g-convex.

# Geodesically Convex Functions

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# Properties of Geodesically Convex Functions

**Theorem**

Let $A \subset M$ be an open g-convex set. Then, a function $f : A \to \mathbb{R}$ is g-convex if and only if it is g-convex in a g-convex neighborhood of every point of $A$.

**Theorem**

Let $A \subset M$ be an g-convex set and let $f : A \to \mathbb{R}$ be a g-convex function. Then, a local minimum point for $f$ is also a global minimum point.

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Let $A \subset M$ be an g-convex set and let $f : A \to \mathbb{R}$ be a g-convex function. Then, a local minimum point for $f$ is also a global minimum point.

# 1st-order Characterization of Geodesic Convexity

## Theorem

Let $A \subset M$ be an open g-convex set, and let $f : A \to \mathbb{R}$ be $C^1$-function. Then, $f$ is g-convex on $A$ if and only if, for every pair of points $x, y \in A$ and an arc-length-parametrized geodesic curve $\gamma : [0, \ell] \to A$ such that $\gamma(0) = x$ and $\gamma(\ell) = y$, we have

$$f(y) - f(x) \geq \nabla f(x)\dot{\gamma}(0)\ell,$$

where $\nabla f(x)$ is the gradient of $f$ at the point $x$.

## Corollary

Let $A \subset M$ be an open g-convex set, and let $f : A \to \mathbb{R}$ be g-convex $C^1$-function. If $\nabla f(x)$ is orthogonal to the tangent space $TM(x)$ of $M$ at $x$, then $x$ is a global minimum point of $f$ on $A$. Furthermore, the set of global minimum points is g-convex.

# 1st-order Characterization of Geodesic Convexity

## Theorem

Let $A \subset M$ be an open g-convex set, and let $f : A \to \mathbb{R}$ be $C^1$-function. Then, $f$ is g-convex on $A$ if and only if, for every pair of points $x, y \in A$ and an arc-length-parametrized geodesic curve $\gamma : [0, \ell] \to A$ such that $\gamma(0) = x$ and $\gamma(\ell) = y$, we have

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# 2nd-order Characterization of Geodesic Convexity

**Theorem**

Let $A \subset M$ be an open g-convex set, and let $f : A \to \mathbb{R}$ be $C^2$-function. Then, $f$ is g-convex on $A$ if and only if, for every point $x \in A$, the following geodesic-Hessian matrix is positive semidefinite

$$H^g f(x) := Hf(x)|_{TM(x)} + |\nabla f_N(x)| B_{\nabla f_N(x)},$$

where $Hf(x)|_{TM(x)}$ is the Hessian of $f$ at the point $x$ restricted to the tangent space $TM(x)$ of $M$ at $x$, and $B_{\nabla f_N(x)}$ is the second fundamental form of $M$ in the normal direction of the vector $\nabla f(x)$.

The matrix vaued function $H^g f$ determines a second-order symmetrical tensor field on $A$.

# Constrained Optimization

## Optimization Problem

Let $f, h_1, \ldots, h_{n-k} : \mathbb{R}^n \to \mathbb{R}$ be $C^m$-functions ($m \geq 1$). Consider the problem $(\mathcal{P})$ described as:

$$\text{Minimize} \quad f(x) \quad \text{with respect to} \quad x \in M,$$

where the equality constraint set $M$ is defined by

$$M := \{x \in \mathbb{R}^n \mid h_1(x) = \cdots = h_{n-k}(x) = 0\}.$$

## Proposition

If $h_1, \ldots, h_{n-k} : \mathbb{R}^n \to \mathbb{R}$ are $C^m$-functions ($m \geq 1$) and the vectors $\nabla h_1(x), \ldots, \nabla h_{n-k}(x)$ are independent for all $x \in M$, then $M$ is a $k$-dimensional Riemannian $C^m$-manifold with the metric structure induced by the Euclidean metric.

# Constrained Optimization

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# The Lagrange Principle, 1st Part

## Lagrange Multiplier Theorem — Necessity

Let $f, h_1, \dots, h_{n-k} : \mathbb{R}^n \to \mathbb{R}$ be $C^1$-functions such that the vectors $\nabla h_1(x), \dots, \nabla h_{n-k}(x)$ are independent at $x = x_0 \in M$. Assume that $x_0$ is a local minimum point for the problem $(\mathcal{P})$. Then there exist multipliers $\mu_1, \dots, \mu_k \in \mathbb{R}$ such that

$$\nabla f(x_0) - \sum_{j=1}^{n-k} \mu_j \nabla h_j(x_0) = 0.$$

If, in addition, $f, h_1, \dots, h_{n-k} : \mathbb{R}^n \to \mathbb{R}$ are $C^2$-functions then, for all $v \in TM(x_0) = \{v \in \mathbb{R}^n \mid \nabla h_1(x_0)v^T = \cdots = \nabla h_{n-k}(x_0)v^T = 0\}$,

$$v^T \left( Hf(x_0) - \sum_{j=1}^{n-k} \mu_j Hh_j(x_0) \right) v \geq 0.$$

# The Lagrange Principle, 2nd Part

## Lagrange Multiplier Theorem — Sufficiency

Let $f, h_1, \ldots, h_{n-k} : \mathbb{R}^n \to \mathbb{R}$ be $C^2$-functions such that the vectors $\nabla h_1(x), \ldots, \nabla h_{n-k}(x)$ are independent at $x = x_0 \in M$. Assume that there exist multipliers $\mu_1, \ldots, \mu_k \in \mathbb{R}$ such that

$$\nabla f(x_0) - \sum_{j=1}^{n-k} \mu_j \nabla h_j(x_0) = 0$$

and, for all
$0 \neq v \in TM(x_0) = \{v \in \mathbb{R}^n \mid \nabla h_1(x_0)v^T = \cdots = \nabla h_{n-k}(x_0)v^T = 0\}$,

$$v^T \Big( Hf(x_0) - \sum_{j=1}^{n-k} \mu_j Hh_j(x_0) \Big) v > 0.$$

Then $x_0$ is a local minimum point for the problem $(\mathcal{P})$.

# Characterization of Geodesic Convexity

## Definition

Let $f, h_1, \dots, h_{n-k} : \mathbb{R}^n \to \mathbb{R}$ be $C^1$-functions such that $\nabla h_1(x), \dots, \nabla h_{n-k}(x)$ are independent for all $x \in M$. Define

$$L(x) := f(x) - \sum_{j=1}^{n-k} \mu_j(x) h_j(x) \qquad (x \in M),$$

where

$$\mu(x)^T := \nabla f(x) \nabla h(x)^T \left( \nabla h(x) \nabla h(x)^T \right)^{-1} \qquad (x \in M).$$

## Theorem

Let $M$ be connected and $f, h_1, \dots, h_{n-k} : \mathbb{R}^n \to \mathbb{R}$ be $C^2$-functions such that $\nabla h_1(x), \dots, \nabla h_{n-k}(x)$ are independent for all $x \in M$. Then $f$ is g-convex on $M$ if and only if, for all $x \in M$, the matrix $H^g L(x)|_{TM(x)}$ is positive semidefinite.

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Let $M$ be connected and $f, h_1, \dots, h_{n-k} : \mathbb{R}^n \to \mathbb{R}$ be $C^2$-functions such that $\nabla h_1(x), \dots, \nabla h_{n-k}(x)$ are independent for all $x \in M$. Then $f$ is g-convex on $M$ if and only if, for all $x \in M$, the matrix $H^g L(x)|_{TM(x)}$ is positive semidefinite.

# Sufficient Condition of Global Optimality

**Theorem**

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$$v^T \Big( Hf(x) - \sum_{j=1}^{n-k} \mu_j(x) Hh_j(x) \Big) v \geq 0.$$

**Corollary**

Assume that the conditions and the statement of the above theorem hold. If, for some $x_0 \in M$,

$$\nabla f(x_0) - \sum_{j=1}^{n-k} \mu_j(x_0) \nabla h_j(x_0) = 0,$$

then $x_0$ is a global minimum point of problem $(\mathcal{P})$.

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