Approximately Convex Functions

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Let D denote a convex set of a real linear space X throughout this talk.

Perturbation of Convex Functions by Bounded Functions

Let $\delta \geq 0$ and let $f : D \to \mathbb{R}$ be of the form $f = g + h$, where $g : D \to \mathbb{R}$ is convex and $\|h\| \leq \delta$, i.e., $|h(x)| \leq \delta$ for all $x \in D$. Then, for all $x, y \in D$ and $t \in [0, 1]$,

$$\begin{aligned} f(tx + (1-t)y) &= g(tx + (1-t)y) + h(tx + (1-t)y) \\ &\leq tg(x) + (1-t)g(y) + \delta \\ &\leq tg(x) + (1-t)g(y) + t\big[h(x) + \delta\big] + (1-t)\big[h(y) + \delta\big] + \delta \\ &= tf(x) + (1-t)f(y) + 2\delta. \end{aligned}$$

More generally, for all $n \in \mathbb{N}$, for all $x_1, \dots, x_n \in D$ and $t_1, \dots, t_n \in [0, 1]$ with $t_1 + \cdots + t_n = 1$,

$$f(t_1 x_1 + \cdots + t_n x_n) \leq t_1 f(x_1) + \cdots + t_n f(x_n) + 2\delta.$$

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Definition (Hyers and Ulam [HU52])

Let $\delta \geq 0$. Then a function $f : D \to \mathbb{R}$ is called δ-convex if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \delta, \qquad (x, y \in D,\ t \in [0,1]).$$

Theorem (Hyers and Ulam [HU52])

Let X be of finite dimension. Assume that $f : D \to \mathbb{R}$ is δ-convex. Then f is of the form $f = g + h$, where $g : D \to \mathbb{R}$ is a convex function and $h : D \to \mathbb{R}$ is a bounded function such that $\|h\| \leq k_n\delta$, where the positive constant k_n depends only on $n = \dim(X)$.

For the proof, use Helly's theorem.
Hyers and Ulam also proved that

$$k_n \leq \frac{n(n+3)}{4(n+1)} \approx \frac{n}{4}.$$

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On the constant k_n

Green [Gre52], Cholewa [Cho92] obtained much better estimations of k_n showing that, for large n,

$$k_n \leq \frac{\log_2(n)}{2}.$$

Laczkovich [Lac99] compared this constant to several other dimension-depending stability constants and proved that

$$k_n \geq \frac{\log_2(n/2)}{4}.$$

This lower estimate shows that there is no stability results for infinite dimensional spaces X.
A counterexample in this direction was earlier constructed by Casini and Papini [CP93].
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Theorem (Characterization of Bounded Perturbations)

Let $\delta \geq 0$. Then a function $f : D \to \mathbb{R}$ is of the form

$$f = g + h,$$

where $g : D \to \mathbb{R}$ is a convex function and $h : D \to \mathbb{R}$ with $\|h\| \leq \delta$, if and only if, for all $n \leq \dim(X) + 1$, $x_1, \dots, x_n \in D$ and $t_1, \dots, t_n \in [0, 1]$ with $t_1 + \cdots + t_n = 1$,

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Definition

A function $f : D \to \mathbb{R}$ is called δ-Jensen convex if

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} + \delta \qquad (x, y \in D).$$

There is no analogous decomposition theorem for δ-Jensen-convex functions by the counterexample given by Cholewa [Cho92].

However, one can get a Bernstein-Doetsch type regularity theorem:

Theorem (Ng and Nikodem, [NN93])

If $f : D \to \mathbb{R}$ is bounded from above on a nonempty open subset of $H \subset D$ and δ-Jensen convex, then it is 2δ-convex.

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For locally upper bounded δ-Jensen-convex functions, one can obtain the existence of an analogous dimension depending stability constant j_n (defined similarly as k_n above). The sharp value of j_n was found by Dilworth, Howard, and Roberts [DHR99] who proved that

$$j_n = \frac{1}{2}\left([\log_2(n)] + 1 + \frac{n}{2^{[\log_2(n)]}}\right) \leq 1 + \frac{1}{2}\log_2(n)$$

is the best possible value for j_n.
(Here $[\cdot]$ denotes the integer-part function).

The connection between δ-Jensen-convexity and δ-$\mathbb{Q}$-convexity was investigated by Mrowiec [Mro01].

If $D \subset \mathbb{R}$ and the δ-convexity inequality is supposed to be valid for all $x, y \in D$ except a set of 2-dimensional Lebesgue measure zero then one can speak about *almost δ-convexity*. Results in this direction are due to Kuczma [Kuc70] (the case $\delta = 0$) and Ger [Ger88] (the case $\delta \geq 0$).

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Perturbation of Convex Functions by p-Lischitz Functions

Let $\varepsilon \geq 0$, $p \in (0,1]$, and let $f : D \to \mathbb{R}$ be of the form $f = g + h$, where $g : D \to \mathbb{R}$ is convex and $h : D \to \mathbb{R}$ is (ε, p)-Lipschitz, i.e.,

$$|h(x) - h(y)| \leq \varepsilon \|x - y\|^p \qquad (x, y \in D).$$

Then, for all $x, y \in D$ and $t \in [0,1]$,

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Various Definitions, Recent Research Directions

A function $f : D \to \mathbb{R}$ satisfying, for some $C \in \mathbb{R}$

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + C\big(\|x-y\|\big)^p$$

is called *p-paraconvex* (cf. Rolewicz [Rol80, Rol79, Rol80b, Rol80a, Rol80c, Rol00, Rol01, Rol02, Rol05, Rol99];
More generally, f is called α-paraconvex if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + \alpha\big(\|x-y\|\big).$$

Recently, Luc, Ngai, and Théra [LVNT00, VNLT00] have introduced the following notion: f is called approximately convex, if for all $x_0 \in D$, for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + \varepsilon\|x-y\|$$

if $x, y \in B(x_0, \delta)$.

Definition

Let $\varepsilon, \delta \geq 0$ be constants. A function $f : D \to \mathbb{R}$ is called (ε, δ)-convex if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon t(1-t)\|x-y\| + \delta$$

for $x, y \in D$, $t \in [0, 1]$.

It is not difficult to see that if f is of the form $f = g + h + \ell$, where
— $g : D \to \mathbb{R}$ is convex,
— $h : D \to \mathbb{R}$ is bounded with $\|h\| \leq \delta/2$,
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Theorem ([Pál03])

Let $I \subset \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ and ε, δ be nonnegative numbers. Then the following conditions are pairwise equivalent:

(i) f is (ε, δ)-convex on I, i.e., for all $x, y \in I$, $t \in [0, 1]$.

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + \varepsilon t(1-t)|x-y| + \delta$$

(ii) For $x, u, y \in I$ with $x < u < y$,

$$\frac{f(x) + \delta - f(u)}{x - u} \le \frac{f(y) + \delta - f(u)}{y - u} + \varepsilon;$$

(iii) There exists a function $p : I \to \mathbb{R}$ such that, for $x, u \in I$,

$$f(u) + p(u)(x-u) \le f(x) + \frac{\varepsilon}{2}|x-u| + \delta;$$

(iv) If $x_1, \dots, x_n \in I$, $t_1, \dots, t_n \ge 0$, $t_1 + \dots + t_n = 1$ and $u := t_1x_1 + \dots + t_nx_n$, then

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$$f(u) \le t_1 f(x_1) + \cdots + t_n f(x_n) + \frac{\varepsilon}{2}\Big(t_1|x_1-u| + \cdots + t_n|x_n-u|\Big) + \delta.$$

Proof of (i)⇒(ii): Assume that f is (ε, δ)-convex and let $x < u < y$ be in I. Choose $t \in [0, 1]$ such that $u = tx + (1 - t)y$, that is let $t = (y - u)/(y - x)$. Then by the (ε, δ)-convexity of f, we get

$$f(u) \le \frac{y-u}{y-x} f(x) + \frac{u-x}{y-x} f(y) + \varepsilon \frac{(y-u)(u-x)}{y-x} + \delta,$$

which is equivalent to (ii).

Proof of (ii)⇒(iii): Assume that (ii) holds and define

$$p(u) := \sup_{x \in I,\, x < u} \left(\frac{f(x) + \delta - f(u)}{x - u} - \frac{\varepsilon}{2} \right) \qquad u \in I.$$

Then, due to (ii), we have

$$\frac{f(x) + \delta - f(u)}{x - u} - \frac{\varepsilon}{2} \le p(u) \le \frac{f(y) + \delta - f(u)}{y - u} + \frac{\varepsilon}{2}$$

for all $x < u < y$ in I. The left inequality yields (iii) if $x < u$, and analogously, the right inequality reduces to (iii) if $x > u$. The case $x = u$ is obvious.

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for all $x < u < y$ in I. The left inequality yields (iii) if $x < u$, and analogously, the right inequality reduces to (iii) if $x > u$. The case $x = u$ is obvious.

Proof of (iii)⇒(iv): To deduce (iv) from (iii), let $x_1, \dots, x_n \in I$, $t_1, \dots, t_n \geq 0$, $t_1 + \dots + t_n = 1$ and $u := t_1 x_1 + \dots + t_n x_n$. Then, substituting x by x_i in (iii), multiplying this inequality by t_i, and adding up the inequalities so obtained, we get

$$\begin{aligned} f(u) &= \sum_{i=1}^{n} t_i[f(u) + p(u)(x_i - u)] \\ &\leq \sum_{i=1}^{n} t_i\Big(f(x_i) + \frac{\varepsilon}{2}|x_i - u| + \delta\Big) = \sum_{i=1}^{n} t_i f(x_i) + \frac{\varepsilon}{2}\sum_{i=1}^{n} t_i|x_i - u| + \delta, \end{aligned}$$

which is the desired inequality.

Proof of (iv)⇒(i): Taking $x_1 = x$, $x_2 = y$, $t_1 = t$, and $t_2 = 1 - t$ in condition (iv), one can see that the inequality (iv) reduces to (i).

Theorem (First decomposition) ([Pál03])

Let $f : I \to \mathbb{R}$ and $\varepsilon, \delta \geq 0$. Then f is (ε, δ)-convex if and only if there exists an $(\varepsilon, 0)$-convex function $\phi : I \to \mathbb{R}$ such that $\|f - \phi\| =\leq \delta/2$.

The proof of the implication $\Rightarrow$ is easy. To prove the converse, assume that f is (ε, δ)-convex and apply the previous theorem. Then there exists a function $p : I \to \mathbb{R}$ such that, for all $x, u \in I$,

$$f(u) + p(u)(x - u) \leq f(x) + \frac{\varepsilon}{2}|x - u| + \delta;$$

Define, for $x \in I$,

$$\phi(x) := \sup_{u \in I}\Big(f(u) + p(u)(x - u) - \frac{\varepsilon}{2}|x - u| - \frac{\delta}{2}\Big)$$

Then we have that $\phi(x) \leq f(x) + \delta/2$ for all $x \in I$.
On the other hand, $f(x) - \delta/2 \leq \phi(x)$. Thus, $\|f - \phi\| \leq \delta/2$.

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Definition

A function $p : I \to \mathbb{R}$ is called ε-nondecreasing if, for all $x \le y$ in I,

$$p(x) \le p(y) + \varepsilon.$$

Theorem ([Pál03])

Let $I \subset \mathbb{R}$ be an open interval and $p : I \to \mathbb{R}$. Then p is ε-nondecreasing if and only if there exists a nondecreasing function $q : I \to \mathbb{R}$ such that $\|p - q\| \le \varepsilon/2$.

Proof of $\Rightarrow$. Assume that q is nondecreasing such that $\|p - q\| \le \varepsilon/2$. Then for $x \le y$, we have

$$\begin{aligned} p(x) &\le q(x) + |p(x) - q(x)| \le q(y) + \frac{\varepsilon}{2} \\ &\le p(y) + \frac{\varepsilon}{2} + |p(y) - q(y)| \le p(y) + \varepsilon. \end{aligned}$$

Thus, p is ε-nondecreasing.

Proof of $\Leftarrow$. Conversely, assume that p is ε-nondecreasing and define

$$q(x) := \sup_{x \in I,\, x \geq v} \left(p(v) - \frac{\varepsilon}{2} \right) \qquad (x \in I).$$

Then q is obviously nondecreasing. By its definition, we have that

$$p(x) - \frac{\varepsilon}{2} \leq q(x).$$

On the other hand, using that p is ε-nondecreasing, $p(v) \leq p(x) + \varepsilon$ for all $v \leq x$, whence

$$q(x) = \sup_{x \in I,\, x \geq v} \left(p(v) - \frac{\varepsilon}{2} \right) \leq p(x) + \frac{\varepsilon}{2}.$$

The two inequalities obtained yield that $\|p - q\| \leq \varepsilon/2$.

Corollary ([Pál03])

Let $\phi : I \to \mathbb{R}$ be an $(\varepsilon, 0)$-convex function on I, where $\varepsilon \geq 0$. Then there exists an increasing function $q : I \to \mathbb{R}$ such that

$$\phi(u) + q(u)(x-u) \leq \phi(x) + \varepsilon|x-u| + \delta \qquad (x, u \in I).$$

By a previous theorem, there exists a function $p : I \to \mathbb{R}$ such that

$$\phi(u) + p(u)(x-u) \leq \phi(x) + \frac{\varepsilon}{2}|x-u| \qquad (x, u \in I).$$

Interchanging x and u and adding up the two inequalities, we get

$$(p(u) - p(x))(x-u) \leq \varepsilon|x-u| \qquad (x, u \in I).$$

If $x < u$, then $p(x) - p(u) \leq \varepsilon$, whence p is ε-nondecreasing. By the previous result, there exists an increasing function $q : I \to \mathbb{R}$ such that $\|p - q\| \leq \varepsilon/2$. Thus, for all $x, u \in I$, we get

$$\begin{aligned}\phi(u) + q(u)(x-u) &\leq \phi(u) + p(u)(x-u) + |(p(u) - q(u))(x-u)| \\ &\leq \phi(u) + p(u)(x-u) + \frac{\varepsilon}{2}|x-u| \leq \phi(x) + \varepsilon|x-u|.\end{aligned}$$

Theorem ([Pál03])

Let $\phi : I \to \mathbb{R}$ and $\varepsilon \geq 0$. Then there exists an increasing function $q : I \to \mathbb{R}$ such that, for $x, u \in I$,

$$\phi(u) + q(u)(x - u) \leq \phi(x) + \varepsilon|x - u|.$$

if and only if there exists a convex function $g : I \to \mathbb{R}$ such that $\ell := \phi - g$ is ε-Lipschitz.

Proof of $\Leftarrow$: Assume that $\phi = g + \ell$, where g is convex and ℓ is ε-Lipschitz. Then, there exists an increasing function $q : I \to \mathbb{R}$ such that

$$g(u) + q(u)(x - u) \leq g(x) \qquad (x, u \in I).$$

The function ℓ also satisfies

$$\ell(u) \leq \ell(x) + \varepsilon|x - u| \qquad (x, u \in I).$$

Adding up these inequalities, we get that ϕ satisfies

$$\phi(u) + q(u)(x - u) \leq \phi(x) + \varepsilon|x - u| \qquad (x, u \in I).$$

Proof of $\Rightarrow$: Conversely, assume that

$$q(u)(x-u) \le \phi(x) - \phi(u) + \varepsilon|x-u| \qquad (x, u \in I).$$

Define now $g : I \to \mathbb{R}$ by $g(x) := \int_{x_0}^{x} q$, where x_0 is a fixed element of I.

Then, q being nondecreasing, we get that g is a convex function.
To complete the proof, we show that $\ell := \phi - g$ is ε-Lipschitz.
For, let $x < y$, $x, y \in I$ be arbitrary. Let $t_0 = x < t_1 < \cdots < t_n = y$ be an arbitrary division of the interval $[x, y]$. Substituting $x := t_{i-1}$, $u := t_i$ for $i = 1, \dots, n$ into the above inequality and adding the inequalities so obtained, we get

$$\sum_{i=1}^{n} q(t_i)(t_{i-1} - t_i) \le \sum_{i=1}^{n} \Big(\phi(t_{i-1}) - \phi(t_i) + \varepsilon(t_i - t_{i-1})\Big)$$
$$= \phi(t_0) - \phi(t_n) + \varepsilon(t_n - t_0) = \phi(x) - \phi(y) + \varepsilon(y - x).$$

Therefore, we obtain

$$g(x) - g(y) \le \phi(x) - \phi(y) + \varepsilon(y - x),$$

that is, $\ell(y) - \ell(x) \le \varepsilon(y - x)$. $\quad\dots\quad$ ℓ is ε-Lipschitz.

Corollary ([Pál03])

Let $\phi : I \to \mathbb{R}$ and $\varepsilon \geq 0$. If there exists a convex function $g : I \to \mathbb{R}$ such that $\ell := \phi - g$ is $\varepsilon/2$-Lipschitz, then ϕ is $(\varepsilon, 0)$-convex on I. Conversely, if ϕ is $(\varepsilon, 0)$-convex on I, then there exists a convex function $g : I \to \mathbb{R}$ such that $\ell = \phi - g$ is ε-Lipschitz.

Theorem ([Pál03])

Let $f : I \to \mathbb{R}$ and $\varepsilon, \delta \geq 0$. If f is of the form $f = g + \ell + h$, where $g : I \to \mathbb{R}$ is convex, $h : I \to \mathbb{R}$ is bounded with $\|h\| \leq \delta/2$, and $\ell : I \to \mathbb{R}$ is $\varepsilon/2$-Lipschitz, then f is (ε, δ)-convex.
Conversely, if f is (ε, δ)-convex on I, then there exist a convex function $g : I \to \mathbb{R}$, a bounded function $h : I \to \mathbb{R}$ with $\|h\| \leq \delta/2$, and an ε-Lipschitz function $\ell : I \to \mathbb{R}$ such that $f = g + h + \ell$.

Open Problem

What happens if $f : D \subseteq X \to \mathbb{R}$ and D is not one dimensional?

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